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Guidelines in mathematics education

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Inspired teaching at all levels, from the kindergarten through the graduate school, is a necessary condition for reform in mathematics education.

THE NATIONAL Council of Teachers of Mathematics was born in Cleveland, Ohio, in February, 1920. Its stated purpose was "to assist in promoting the interests of mathematics in America, especially in the elementary and secondary fields." For approximately forty years it has been faithful to this purpose, and during these four decades its services to both elementary and secondary teachers of mathematics have steadily increased. We have a publications program unequaled by any organization in its potentialities for providing professional assistance to its membership. Our three journals have been instrumental in turning many a dull classroom into a center of intellectual activity. Our twenty-five yearbooks provide the finest professional mathematics library available, and our supplementary publications have served to enrich mathematics programs on all levels of instruction.

Our Washington office was established for no other purpose than to be of increasing service to our members, and the ever-expanding volume of business so graciously handled by the office staff is adequate evidence that it has effectively served that purpose. The steady increase in membership reflects the confidence of mathematics teachers in our organization. Not only has the membership increased, but the rate of increase tells a significant story. We were gratified with the 17 per cent increase in 1956–57, but we did even better in the next two years, when the respective increases were 20 per cent and 22 per cent. During 1959–60, approximately eight thousand new names have been added to our membership list, a 31 per cent increase, and by May 1, 1960, the total number of members and subscribers was about 31,500.

We look in perspective on forty years of history, and it requires no penetrating analysis of our record to question the accuracy of the oft-quoted statement that "life begins at 40." The currents of intellectual life permeate the entire history of the Council, and they were never more in evidence than at our thirty-eighth annual meeting in Buffalo, New York. But life begets life, and, as we begin our forty-first year of service "in promoting the interests of mathematics in America," we should indeed give particular attention to the life of our fifth decade which does begin at 40.

We have the largest membership in history. Our 73 Affiliated Groups provide a source of great potential strength. Mathematics teachers again have developed a large measure of self-respect. Across the country they have come to realize that they have a vital contribution to make to the basic education of our young people, and along with their administrators and supervisors they look to the National Council for leadership. Through four dec-

ades of history we have never operated in an educational climate so friendly to the growth and development of mathematical ideas in the classrooms of America, and from that viewpoint we may well consider that life could indeed "begin at 40." The use of the word "could" is not an accident. I use it with deliberate purpose, for the future of the National Council "in promoting the interests of mathematics in America, especially in the elementary and secondary fields" can lead to a more abundant professional life, but it can also lead to professional obscurity.

What guidelines should be followed if the very healthy life of yesterday is to beget an even healthier life for tomorrow? Reflecting the outcomes of our Policy Conference in Chicago, reported in the April issue of THE MATHEMATICS TEACHER, may I now suggest five such guidelines for your thoughtful consideration:

1. *With respect to the curriculum*

There has probably never been a time in the long history of mathematics education when the curriculum has received more attention than at present. Significant curricular studies have either been completed or are now in process. Ambitious school administrators, hearing of the "new mathematics" from friendly but frequently unreliable sources, expect their mathematics teachers to discard a faded but familiar program pattern on Friday afternoon and appear on Monday morning professionally resplendent in a new curricular design. The administrators themselves rarely understand the nature of the changes they are advocating, nor are the mathematics teachers, in most cases, ready or qualified to make them.

It is in such situations that teachers become restless, uncertain, insecure, and frequently frustrated. Thousands of letters reaching our Washington office testify to the fact that they look to the National Council of Teachers of Mathematics for guidance; it is our responsibility to provide it. The Council should sponsor no given curriculum, but it should provide critical reviews and appraisals of all mathematics curricula proposed for the elementary and secondary schools. To use the professional competence which resides in our membership to interpret new developments; to analyze current curricular projects, emphasizing their common elements as well as their significant differences: and to review new instructional materials is to provide a service which will simultaneously answer the call of our membership and increase our professional stature. It is a service long overdue.

2. *With respect to mathematics in the elementary school*

Any organization dedicated to the improvement of mathematics education must be concerned with mathematics in the elementary school, for mathematics education is a process which is continuous and indivisible. The development of a mathematician, an engineer, or a teacher of mathematics does not begin in the university. There are those who say that it begins in the high school, but one might well argue that it begins in the elementary school, for the curriculum choices made in the high school are largely influenced by attitudes and interests developed in the elementary grades. This important principle is reflected in the creation of the University of Illinois Arithmetic Project and in the extension of the program of the School Mathematics Study Group to include the mathematics of the elementary school.

The National Council of Teachers of Mathematics has recognized its responsibility to mathematics instruction in the elementary school. The Twenty-fifth Yearbook, released at the annual meeting in Buffalo, is the third of the series dedicated entirely to the teaching of arithmetic. *The Arithmetic Teacher* was established better to serve the interests of the elementary teacher, and at each of our professional meetings there are sections concerned with the problems of mathe-

matics teaching in the elementary school. But the far-reaching significance of the insights and understandings established in these early years calls for even greater effort, and one of the recommendations of the Chicago Policy Conference is that "the National Council of Teachers of Mathematics put as much emphasis on elementary education as on secondary teaching." While I am in complete sympathy with the spirit of this recommendation, the proposed equality of emphases is based on an assumption of curricular divisibility which I reject. To match a pound of emphasis on the secondary level with a pound of emphasis on the elementary level may appear to balance the scales, but to place our faith in such mechanics is to ignore the organic structure of the mathematics curriculum. The foundations of mathematical power *can* be established in the elementary school, and no one questions the significance of the mathematics instruction received during these early years. Elementary schools call for the finest kind of service we can provide, but any measure of emphasis on this level which yields understanding and insight is distributed manyfold throughout the long educational continuum.

3. *With respect to teacher certification*

Our dedication to the improvement of mathematics education in America throws the spotlight, not only on the curriculum, but also on the classroom teacher. The quality of the curriculum is important, but no mathematics program will ever be any better than the faculty responsible for it. The curriculum is not a disembodied force which in some unique and mysterious manner moves through the classrooms of America, stirs the imagination of our students, enriches their mathematical insights, and develops their highest potential. To achieve such desirable outcomes a teacher is needed, and the real curriculum includes those methods and procedures by which he brings meaning and significance to the mathematical content covered. No student will be guided toward an understanding of mathematical method through teaching procedures which feast his memory and starve his reason. The beauty of mathematical structure will be forever denied to those who continue to sit in classrooms where mathematics is taught only as a tool subject and routine drill is emphasized. A poor curriculum translated into practice through inspired teaching is, in fact, more to be desired than a good curriculum in the hands of dull and unimaginative teachers.

"A new curriculum," says Dr. Albert E. Meder, Jr., "is no substitute for creative teaching. Indeed, it is the other way about. Creative teaching would be a very satisfactory substitute for a new curriculum."

If we are serious in our purpose of improving the teaching of mathematics in America, we must then be concerned with the quality of the classroom teacher in both elementary and secondary schools, which means that we must be concerned with teacher education and certification. This concern is shared by other professional organizations, and our co-operative effort is essential in any satisfactory solution to this important problem. Funds have been provided by the Carnegie Corporation for the support of a study concerned with certification requirements for teachers of science and mathematics. This study was proposed by the National Association of State Directors of Teacher Education and Certification, an unincorporated organization, and the funds are to be administered by the American Association for the Advancement of Science. One of the four goals of this eighteen-months study is:

> To develop procedures whereby *representatives of logically interested organizations* and state certification and accreditation officials may work together effectively in the development of teacher preparation programs, and to prepare suggested guides for program approval by state certification officers.

It is hoped that in developing these "procedures" the National Council of

Teachers of Mathematics may be recognized as one of the "logically interested organizations." It should be pointed out, however, that this study is limited to teachers of secondary school science and mathematics. The need is great on the secondary level, but it is even greater on the elementary level, and the Council is concerned with both levels.

The great debate with respect to the relative values of subject matter and professional courses in a teacher education program is wasteful and unproductive. The creative teacher, called for by Dr. Meder, is not the product of a program reflecting any such dichotomy, and our co-operative efforts may be much more fruitful if we operate on the principle that *all preparation is professional.* Fifty-three years ago a dull but devoted teacher in a Canadian school said to me, "Harold, you don't know enough," and he invited me to remain during that first delightful hour of release following the close of the school day and write "Knowledge is power" three hundred times. He was right when he said I didn't know enough, and he would be right if he were alive to say it today. But *he* didn't know enough either. He wanted to impress me with the fact that "knowledge is power," and he used a method which only served to defeat his very purpose. He did demonstrate his "power" to hold in bondage for an agonizing hour the body of a rebellious student, but that was a result of ignorance and not of knowledge.

I do not accept the general proposition that mathematical knowledge alone yields teaching power, for every undergraduate student has suffered from counterexamples which disprove it. Albert Einstein, in the later years of his fruitful and productive life, wrote that "knowledge is dead; the school serves the living," which I interpret to mean that, while mathematical knowledge is a necessary condition to teaching power, it is not a sufficient condition. The creative teacher must have large mathematical insights, but he must also have large human insights.

4. *With respect to the college teaching of mathematics*

The mathematics classrooms of America are calling for teachers who not only *know* mathematics, but who also know how to guide their students as together they travel the educational continuum leading to growing mathematical maturity. To answer this call is a major concern of the National Council, and since teachers tend to teach as they, themselves, have been taught, the quality of mathematics teaching in college classrooms is one of our major concerns. We are grateful for those dedicated professors whose teaching genius stirs the intellectual qualities of their students, awakens their latent potentialities and opens inviting channels of investigation and study. From such classrooms come the great mathematics teachers of America.

Buildings are essential. Materials and facilities are of real value. Modern teaching appliances are helpful, but nothing can surpass the stimulating influence of a devoted and enthusiastic teacher. However, in all too many programs leading to teacher certification in mathematics, the student can complete the requirements in his major field of study without ever coming under the direct and wholesome influence of a real teacher-scholar. Graduate students with little or no interest in teaching, who are somewhat soured by the fact that their assigned teaching load impedes their own research, are all too frequently responsible for undergraduate courses in which sit the mathematics teachers of tomorrow. In some respects they may be helpful, but they can never provide the outreach and the maturity of judgment so valuable to prospective teachers. Mathematics will become important to students when it is important to their teacher. *His* respect for the subject, *his* wide knowledge of its continuing contribution to life, and *his* genuine pleasure in teaching it all "rub off" on the students.

Perhaps the teacher education study proposed by the National Association of State Directors of Teacher Education and Certification should be extended to include the preparation of college teachers. At any rate, it is my considered judgment that if we are to be instrumental in improving the teaching of mathematics in America, "especially in the elementary and secondary fields," we must likewise be concerned with the teaching of mathematics on the college level, and this is particularly true for those of us who have responsibilities in teacher education programs. The men and women now teaching mathematics in the schools of America were *our* students. They sat in *our* classrooms and they listened to *our* lectures. It is *our* attitude toward teaching, it is *our* respect for mathematics, and it is *our* understanding of the role of mathematics in society which "rubs off" on them. Within rapidly passing years their places in our classrooms will be filled by their students, and we will in some measure have before us the tangible results of our own handiwork. We reap no more than we sow, and if we really want to provide the mathematics classrooms of America with qualified, enthusiastic teachers, it is our responsibility to provide these prospective teachers with the kind of teaching that stirs *their* imagination, and increases *their* respect for the teaching profession.

5. *With respect to graduate programs in mathematics*

A beginning teacher faces the responsibilities of his first teaching position with some measure of uncertainty, but with high aspirations and a healthy idealism. He has completed his program of preparation, and now he is ready to guide his students to growing mathematical maturity. Within a few years the actual realities of the classroom have served to increase his uncertainty, his idealism is somewhat tarnished, and a salary structure which rewards professional activity strengthens his feeling that he needs additional study. He recognizes, perhaps for the first time, that college graduation is indeed the commencement of a continuing journey down the long highway of learning leading to professional competence. He must keep up to date with developments in his major field of study. New psychological insights are essential. Curricular problems, quite theoretical when he was a student, suddenly become *his* problems. He recognizes that teaching procedures *do* vary in effectiveness.

He turns to the graduate school for assistance, and while he may get it with respect to curricular problems and teaching procedures, he is all too frequently disappointed with respect to "developments in his major field of study." He wants to follow a program leading to the master's degree in mathematics, for he realizes that knowledge of his field is indeed "a necessary condition to teaching power." He finds, however, that graduate programs in mathematics are, in general, not designed with his needs in mind. They are designed to prepare students for research in mathematics and have little claim on the interests of a secondary school teacher, while at the same time they provide a threat to the quality of his graduate record. There is, in fact, reason to believe, says the Commission on Mathematics, "that the usual graduate course is not suitable, either in content or level of difficulty, because the courses for the secondary teacher should emphasize a broad understanding of the elementary aspects of a field of study and place less emphasis than usual on the mastery of advanced details or complicated proofs." The Commission then points out that "a solution must be found for this problem, since teachers naturally gravitate to other departments for the necessary credit to earn advanced degrees if their special interests are not given careful consideration."

The American Institute of Physics and the American Association of Physics Teachers, in a joint conference on physics

in education held August 8–9, 1959, in New York City, adopted the following resolution, which reflects deep insight into the professional needs of the secondary school teacher:

> Be it resolved that the colleges and universities be urged to strengthen the program in the high schools by encouraging teachers in the secondary schools to take specially designed courses in physics which will bring to them the newer developments in physics and enable them to review the basic concepts and principles of physics. To bring the teacher into such physics classes it is recommended that the courses be organized at the level of the teacher's preparation and with his needs in mind; that arrangements be made with the Graduate School and Departments of Education which will permit the use of credit in these courses toward the master's degree in education.

But the physicists have no corner on the intellectual nourishment responsible for the professional insights reflected in this resolution. It is available to all, and our Departments of Mathematics, deeply concerned about the improvement of mathematics teaching on the secondary level, could take a long and fruitful step toward the achievement of this highly desirable purpose by providing "specially designed courses in *mathematics* which will bring to them (teachers in the secondary schools) the newer developments in mathematics and enable them to review the basic concepts and principles of mathematics." The voice of the National Council speaks for mathematics teachers in the secondary schools of America and asks that such courses be made available as a part of any graduate program designed to improve teachers' professional competence.

Throughout the years men and women of insight have always recognized mathematics as an instrument of great educational value. More recently their number appears to have increased. Friends of mathematics have appeared from all directions, and the number of people who have suddenly awakened to the fact that mathematics permeates the very fabric of our culture is indeed surprising. Substantial financial support has become available. The cash registers of industry are open to those who seek funds for mathematical study. Foundations look with favor on proposals designed to upgrade the teaching of mathematics, and even Congress, recently concerned with the length of the inch, is providing large sums of public money for the improvement of the mathematics curriculum and for the in-service education of mathematics teachers.

But the day will come in the not-too-distant future when these financial sources will no longer be available. To provide for that day let the 73 Affiliated Groups join with the National Council of Teachers of Mathematics in a unified effort to maintain and even increase the professional momentum of the past five years. To such a challenge the National Council responds in the modified words of "The Brook" as it flows steadily forward to join the "brimming river":

> For funds may come,
> And funds may go,
> But we go on forever.

"Automation will not invade every field of activity. Economic considerations, rather than feasibility of applying the techniques of automation, will be limiting factors. But any consideration of the culture of tomorrow must involve an evaluation of the impact of automation."—*E. T. Welmers*, "Instruction in Arithmetic," Twenty-Fifth Yearbook of the National Council of Teachers of Mathematics.

New thinking in mathematical education

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Before reform can take place in mathematics education, reformers must consider the questions posed in this article.

THIS PAPER is concerned only with the statement of problems and not with their solutions. Every care has been taken in the following only to raise issues, and not to influence presentations and discussions toward one or another point of view.

We are concerned with reaching conclusions and making decisions on a program of mathematical education that is mathematically sound, societally important, and pedagogically feasible for our time. The assignment to teachers and educators is thus clear. We have heard and are hearing about mathematics, both pure and applied, that is considered by mathematicians to be important today. We also hear of new thinking in mathematics that has importance for the teaching of elementary and secondary school mathematics. What mathematics is, and what mathematics is considered important, are rightfully the decisions of mathematicians. What portion of this mathematics can be taught below university level, to whom it can be taught, and the way it can be taught is then our problem.

The assignment to school people becomes the following:

a) Looking at existing programs and examinations in mathematics in the elementary and secondary schools, to ask, in the light of modern developments in mathematics, if the content is appropriate for the education of youth in this day.
b) To decide the nature of the mathematics that all the capable youth should learn if they go on to further study of science, engineering, and mathematics in the university. To find out what mathematical training and competence the university professor would desire of his beginning students.
c) To discover, if possible, in view of the shortage in many countries of the scientifically-trained personnel needed in industry, government, research, and teaching, how mathematics can be presented so as to attract and produce a larger number of secondary school graduates with quality learning in the subject.
d) To make such recommendations for modifications, additions, deletions, and improvements of existing programs so that responsible authorities and mathematicians can influence change as they see the need for it.

The above assignment is predicated on the need for reform. However, before we proceed, one should investigate and be convinced of the need for a change. A good program should not be changed merely because some persons are dissatisfied with it. The activities of the past ten to fifteen years, when brought to a head, give sufficient evidence that consideration of reform, if indeed not reform itself, is now in order. We briefly mention a few considerations:

a) The new developments in graduate and research mathematics imply a necessary shift in emphasis for secondary school mathematics. New thinking in mathematics tells us much about abstract algebra, topology, vector spaces, structures, theory of sets, and so on, all of which indicate a necessary change in point of view on what mathematics is today.
b) The new applications of mathematics signify new problem material. Probability, statistical inference, finite mathematical structures, game theory, linear programming, numerical analysis, automation—all indicate a change in the applications and useful purposes of mathematics.
c) The development of symbolic logic—and its close relation to mathematics itself in the use of quantifiers, variables of several sorts, sentences, relation, function, and structures—indicate a need for reconsideration of the concepts imbedded in our classical treatment of mathematics.
d) The tremendous increase of knowledge in the various branches of mathematics demands a synthesis and a broader base in what is taught in the preuniversity level. To learn mathematics today requires new, more efficient, more inclusive, and more general approaches. Choquet has made this clear.
e) The changes in the cultural, industrial and economic patterns of many nations call for a basic change in educational patterns. More people must be better trained in scientific knowledge. Even laymen must come to understand science, and, today, knowing mathematics is basic to understanding science.

Thus it is obvious that the case for reform is settled, but what reform is now our task?

Since we are concerned with elementary and secondary (below university) education, we must consider mathematics as it relates to universal education, as well as to the education of those of high scholastic ability. In all our discussions we must consider three educational goals that are overlapping and certainly noncontradictory, namely:

a) Mathematics as liberal education—freedom of the mind.
b) Mathematics as a basis for living and work—as the people's necessary tool.
c) Mathematics as propaedeutics — as foundation for university study.

As we teach mathematics, must we proceed differently to meet each of these goals, or are all goals met by one curriculum, judiciously adjusted to the mental abilities of the students at a given age or grade?

In particular, we are asked to give somewhat specific answers to the following questions:

a) What mathematics should we teach? This implies not only what branches or topics, but the concepts, language, symbolism, and organization of the subject.
b) To whom should we teach mathematics? All students? Only those of scholastically high caliber? Or only those scientifically inclined who will need and use it?
c) How shall we teach our mathematics? As a set of manipulative skills? As a set of ideas? As a set of structures, either intuitively or rigorously developed? As a physical study headed toward an abstract science? Should we stress "doing the mathematics" and "understanding the concepts and development" equally or one more than the other? As algebra, geometry (plane and solid), trigonometry, analysis, etc., or as one continuous development of mathematics?

In the following sections, questions are raised concerning the teaching of the separate branches. The reader will undoubtedly give answers to many of these questions. To others, professional workers must

seek some answers. That there is only one answer or a best answer to any one question is debatable, but it can be hoped that certain agreements on "What," "To Whom," and "How" can emerge, to act as guidelines in any reform that may be contemplated in the mathematical program.

Arithmetic

It is generally agreed that all children should learn how to count, to compute with whole numbers and fractions (both in common and decimal notation), to know common measures, and to use this arithmetic in solving problems. But when to teach this arithmetic formally, through how many years of instruction, and in what manner—that is, by rote repetitions or by understanding and discovery—are unanswered questions. Can children from 5 years to 12 years of age generalize, abstract, and reason sufficiently to do arithmetic as a mathematical science rather than as a computational mechanism?

That arithmetic begins in the physical world of objects is recognized. Today many people (Montessori, Stern, Cuisinnaire, Gattegno, Dienes, Lazar, etc.) are developing materials, some based on color, all purposing to develop meaningful arithmetic. Are these systems necessary for the learning of arithmetic? Is it necessary to build an abstract game right from the start? Do children who have learned the operations on numbers solve problems by applying the operations correctly? If not, how is the arithmetic to be taught so that children do acquire a tool with which to solve quantitative problems? Stated another way, can we teach arithmetic through the use of sets (disjoint) and operations on sets, beginning with physical objects and gradually abstracting concepts, principles, and operational procedures so as to achieve a symbolic theory capable of being used correctly in quantitative situations?

We should know what and how arithmetic should be taught, for this subject is the basis for all later study of mathematics. As the study of mathematics is continued in later school years, to what extent should the study of arithmetic be extended? In particular should numeration to other bases than 10 (for example, the binary system) be introduced? Should modular arithmetic be introduced, and if so, why? How much of the theory of numbers is essential, what particular topics, and how does this development tie in with the study of algebra? What arithmetic knowledge is essential as a foundation for a modern program of secondary school mathematics?

If we can agree on an arithmetic program, is it possible to adapt this program to children of all abilities, the slower children taking longer time to accomplish it? (Shaw's *Pygmalion* indicates patience can be rewarded.) Are rules necessary, e.g., can per cent be taught merely as an application of fractions? Are concepts more important than skills (since we now have all sorts of computation and calculating machines) or are skills more essential?

Algebra

The classical and still common treatment of algebra has been a most mechanical manipulative study. The textbooks and syllabi on algebra are usually devoid of such words as theorem, proof, and laws, but filled with rules and principles. While algebra is presented as a generalized arithmetic of some sort, the use of the concepts—variable, domain, range, relation, function, and transformation—are rarely if ever used. In the light of the great development of abstract algebra over the last thirty years, we should examine critically and carefully our teaching of algebra.

In particular the following questions need investigation and answers before we can successfully reform our instruction:

a) How should the study of algebra be initiated? As generalized school arith-

metic? By study of number systems? Through equations and formulas? Other ways?

b) How should the solution of equations be taught? Through the axioms on operations? Through the use of statement forms, statements, and equivalent equations? Should the quantifiers $\forall$ and $\exists$ be introduced in a first course in algebra?

c) To what extent should inequalities (inequations) be introduced and with what symbolism ($<$, $>$, $\nless$, $\ngtr$, $\neq$, $\leqq$, etc.)? Should the language of sets (union, intersection, solution set, etc.) be used in this treatment? How should absolute value be treated?

d) What should be the treatment of graphs in the study of algebra? Should attention be given to graphs in one dimension, e.g., to illustrate the solution to $x+5>3$? In systems of inequations, should simple problems of linear programming be introduced?

e) How should the extension of the concept of number be treated? Should the negative rationals be attached to the positive fractions and zero, or should the rationals be constructed as ordered pairs of integers? Should isomorphism be introduced? Especially, what treatment should be afforded the real numbers? Should we treat them as infinite decimals, or as sequences or cuts in the rationals?

f) How much structure can be introduced into algebra? Should we emphasize deduction through the use of the commutative, associative and distributive laws, or should we treat algebra in more classical style?

g) To what extent can or should modern abstract algebra be introduced into the secondary school program? In particular should we study groups, rings, fields, matrices, and vector spaces? If this is to be a part of secondary school mathematics, why is it important? What does it contribute?

h) To what extent should sets, that is the language, symbolism, use, and operations of sets, be introduced? What does the use of sets contribute? Should we make use of Cartesian (cross) products, sets of ordered pairs, and equivalence relations? Should a Boolean algebra be taught?

i) What concepts of relation and function are to be treated in secondary school algebra? Is a function always single-valued? What symbolism shall we employ? For example, shall we distinguish $\cos x$ from $\cos (x)$ and $f(x)$ from f?

j) What balance between theory (deduction) and manipulation (the ability to transform equations and fractions and get the correct symbolic result) shall we establish?

Until we reach some agreement on most of these problems we shall be unable to reform the instruction in algebra. What reforms we make should not be predicated on what Russia or other countries do, but on the way we think about algebra as real mathematical development for our students. Finally we must decide for which students we plan our algebra program—all students, university-bound students, technical students, or general students. Does our program of mathematics apply to all students if we adopt it to the maturity level that any given student has reached? Will a reformed program attract more students to mathematical study?

Geometry

That geometry, as physical representation of the world in which we live, must play a larger role in elementary education is now generally accepted. That an intuitive physical geometry is preparatory to the study of geometry as a deductive science is also generally accepted. But the nature of the treatment of deductive geometry at the secondary school level is perhaps the most controversial issue in mathematical education today. Euclid, with all its defects, has held the stage in secondary school mathematics for the

past two hundred years. Now, however, with the defects of the usual Euclidean synthetic geometry well known, and with the advent of many ways of treating this geometry correctly at hand, we are still in a quandary. The rigorous treatments of Hilbert, Veblen, and others are too difficult for a first approach to deductive geometry. What shall we do?

a) Keep the present system? We could include a few more dependent axioms on order and separation to be more complete in our deductions, but the stress could be on discovery, deduction, and not on axiomatics.
b) Treat solid synthetic geometry deductively or do most of solid geometry intuitively as an extension of plane geometry properties to three-space?
c) Do an algebraic geometry, assuming the real numbers and their properties and starting with the affine line? This would presuppose some maturity in algebraic thinking.
d) Use a modified version of the Erlanger program of Felix Klein, where we can begin with constructions involving the possible transformations preserving metric properties, and then proceed more rigorously to the usual presentation of plane geometry?
e) Use several geometries, in a deductive form, but not with airtight axiomatic systems? For example, we could do the usual Euclidean geometry, followed by analytic geometry, and then a vector geometry.
f) If vectors are introduced, how should this be done, and to what extent can we develop the vector space employed?
g) Does descriptive geometry still have a place in secondary school mathematics? If so, to what extent? What purpose does it serve, if any?
h) Should non-Euclidean (hyperbolic, elliptic) geometry be introduced into the secondary school program? If so, why?
i) Should projective geometry be taught in the secondary school?

The answers to these questions are in part predicated on the geometry program of the university, which is also in a state of uncertainty. Geometry has been a subject neglected by collegiate and graduate mathematicians (unless topology be considered in this category). Nevertheless, geometry has always been one of the great liberalizing subjects of instruction, and the one subject where logic and its uses has loomed large. If arithmetic, algebra, and analysis are to employ logic and deduction, the role of geometry in the total mathematical education program takes on a different aspect than hitherto.

Trigonometry

Most countries include very early in their mathematical programs an introduction to the trigonometric ratios, sine, cosine, and tangent for the acute angles of a right triangle. Most persons admit that the value of trigonometry as a method of solving triangles, especially with the use of logarithms, is *small* today compared with the use of the subject with respect to the periodic nature of its functions. The questions confronting us are the following:

a) To what extent should trigonometry be developed as the theory of *angles*, through the use of co-ordinates?
b) Should the subject be developed purely as functions of real numbers, either by using the length of arc of a unit circle as the argument with the co-ordinates of the terminal point (abscissa and coordinate) as values of the argument (the so-called wrapping function), or by using complex numbers and the complex plane?
c) Should the cos $[x-y]$ be the key function in developing trigonometric analysis?
d) What applications of trigonometry should be stressed—equations, identities, periodicity—applied to sound, light, electricity, or solution of triangles, including spherical geometry?

e) Should trigonometry be treated as a special subject or included as a part of algebra and geometry and later through the use of series as analysis?

Analysis

In most countries, students preparing for the university study a substantial amount of analytic geometry and calculus in the secondary school. This is not true of all countries. Assuming that analysis, to some extent, belongs as a part of secondary school mathematics, we should determine the type, and the degree to which it should be introduced. Since this subject is usually reserved for better students, our problems are not as difficult as in the other secondary school branches. Yet the modernization of treatment of these subjects raises a real point of issue. To what extent can we be more than intuitive and short of the rigorous in our presentation? The following questions suggest themselves for consideration.

a) What treatment should be given to limits and continuity? Are δ and ϵ methods appropriate? What degree of rigor is necessary?
b) What particular functions—algebraic, exponential, logarithmic, trigonometric, and *inverse*—should be treated, and with what symbolism and concept?
c) To what extent should the development be based on sets and structure? To what extent should the treatment be a first-level naïve approach?
d) Should both differential and integral calculus be introduced early? If so, what should be the initial treatment of integral calculus, the inverse of differentiation, or summation? How should both these calculi be related, through the fundamental theorem or merely by intuitive methods?
e) Should analytic geometry enter high school as a separate geometry course, or be treated as a part of the analysis course through the use of calculus?
f) If analytic geometry should be done (either as geometry or with the calculus) should it be by the use of vectors? How far should the vector treatment of the conics be carried?
g) To what extent should set functions and their application to probability be studied? What reform should we give to the teaching of combinatorial analysis, if any?
h) Should statistical inference be taught at the secondary level? In what manner? To what extent?

Conclusion

The answers to the foregoing questions are not sufficient to establish a complete program in mathematics. Some groups now working on curricular reform may have discussed all the above points, but they certainly have not directly communicated their answers to the teachers at large. It is necessary for all teachers to consider these questions. Further queries will no doubt loom important to many readers. However, the questions raised above should tend to focus our deliberations so that we see issues of agreement and dissent, and come to certain conclusions with regard to:

a) What specific mathematics should and can we teach, that is, what concepts, topics, symbolism, language, and problems (applications) at the preuniversity level?
b) What specific objectives do we have in mind in deciding on specific mathematical development for students at the several levels of ability?
c) How should we organize and present (reform) our mathematics to make it conform to the mathematics considered important today?
d) How can we communicate our purpose and program to the community at large?

Applications of finite arithmetic, II

ROY DUBISCH, *Fresno State College, Fresno, California.*
An application of finite arithmetic to the problem of coding and uncoding messages is presented here.

IN OUR PREVIOUS article on this topic[1] we explored the applications of finite arithmetic to factoring. In this article we will utilize finite arithmetic together with 2×2 matrices to develop a simple method of coding messages. This coding method is not only easy to use but, since it destroys frequencies, it is also very difficult to decipher messages coded by this method without the key.

As is the case with finite arithmetics, extensive descriptions of matrices are easily available to the reader.[2] Hence we will only summarize the necessary definitions here. It is to be assumed in all that follows that a_{11}, a_{12}, a_{21}, a_{22}, b_{11}, b_{12}, b_{21}, b_{22} are the numbers of some finite arithmetic A_p where p is a prime number. (See previous article for a brief discussion of A_p.)

Definition 1: A 2×2 matrix with elements in A_p is a square array of numbers of A_p represented by

$$A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Definition 2:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

if and only if $a_{11}=b_{11}$, $a_{12}=b_{12}$, $a_{21}=b_{21}$, and $a_{22}=b_{22}$.

Definition 3:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} =\begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix}.$$

Definition 4: If

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and A and B are 2×2 matrices such that $AB=BA=I$ we say that B is the inverse of A.

THEOREM: If $D=a_{11}a_{22}-a_{12}a_{21}\neq0$, the matrix A has the inverse

$$\begin{pmatrix} a_{22}D^{-1} & -a_{12}D^{-1} \\ -a_{21}D^{-1} & a_{11}D^{-1} \end{pmatrix}.$$

Let us now take the elements of A_{29} to be $1, -1, 2, -2, 3, -3, \cdots\cdots, 14, -14$, and 0 where we use the fact that $-1=28$ modulo 29 (since $-1-28$ is divisible by 29); $-2=27$ modulo 29, $\cdots$, $-14=15$ modulo 29. (We use these negative numbers simply because it is easier, for example, to multiply -1 by -2 than 28 by 27.) Next we establish the cipher alphabet as in the table that follows:

[1] THE MATHEMATICS TEACHER, LIII (May, 1960), 322–324.

[2] See, for instance, Arnold Wendt, "A Simple Example of a Noncommutative Algebra," THE MATHEMATICS TEACHER, LII (1959), 534–540, or any text on modern algebra or matrix theory.

A	B	C	D	E
1	-1	2	-2	3
F	G	H	I	J
-3	4	-4	5	-5
K	L	M	N	O
6	-6	7	-7	8
P	Q	R	S	T
-8	9	-9	10	-10
U	V	W	X	Y
11	-11	12	-12	13
Z	,	.	–	
-13	14	-14	0	

Suppose now that we have the message to code, "Join the NCTM now." For the convenience of the person who will later decode the message let us rewrite this message with dashes (–) between the words. Then we put underneath each symbol the corresponding element of A_{29} according to our table. This is shown below where, also, the message is marked off in groups of 4 (with dashes added to provide a last group of 4).

J	O	I	N	–	T	H	E
-5	8	5	-7	0	-10	-4	3
–	N	C	T	M	–	N	O
0	-7	2	-10	7	0	-7	8
W	.	–	–				
12	-14	0	0				

Next we write each group of 4 elements of A_{29} as a matrix going from left to right and from top to bottom. Thus we obtain the 5 matrices

$$A_1=\begin{pmatrix}-5 & 8\\ 5 & -7\end{pmatrix},\quad A_2=\begin{pmatrix}0 & -10\\ -4 & 3\end{pmatrix},$$

$$A_3=\begin{pmatrix}0 & -7\\ 2 & -10\end{pmatrix},\quad A_4=\begin{pmatrix}7 & 0\\ -7 & 8\end{pmatrix},$$

$$A_5=\begin{pmatrix}12 & -14\\ 0 & 0\end{pmatrix}.$$

Now we withdraw from the heavily guarded vaults our "key" matrix (which may be changed from time to time). Let us suppose that today it is

$$B=\begin{pmatrix}-1 & 1\\ 2 & 1\end{pmatrix}.$$

Then we calculate $BA_1, BA_2, \cdots, BA_5$ to obtain, by the use of definition 3,

$$BA_1=\begin{pmatrix}10 & -15\\ -5 & 9\end{pmatrix},\quad BA_2=\begin{pmatrix}-4 & 13\\ -4 & -17\end{pmatrix},$$

$$BA_3=\begin{pmatrix}2 & -3\\ 2 & -24\end{pmatrix},\quad BA_4=\begin{pmatrix}-14 & 8\\ 7 & 8\end{pmatrix},$$

$$BA_5=\begin{pmatrix}-12 & 14\\ 24 & -28\end{pmatrix}.$$

Since some of our numbers are not in the list $1, -1, \cdots, 14, -14, 0$, we use the concept of equality modulo 29 to rewrite these matrices as

$$BA_1=\begin{pmatrix}10 & 14\\ -5 & 9\end{pmatrix},\quad BA_2=\begin{pmatrix}-4 & 13\\ -4 & 12\end{pmatrix},$$

$$BA_3=\begin{pmatrix}2 & -3\\ 2 & 5\end{pmatrix},\quad BA_4=\begin{pmatrix}-14 & 8\\ 7 & 8\end{pmatrix},$$

$$BA_5=\begin{pmatrix}-12 & 14\\ -5 & 1\end{pmatrix}.$$

Next we list the numbers in these matrices in the order used in obtaining the original matrices and, underneath each number, list the equivalent symbol. The result: S, JQHYHWCFCI.OMOX,JE is then sent to the desired person. Note that, as promised, frequencies are destroyed in the coding. In our message there were three "N's" which appear, in order, as "Q," "E," and "M" in the coded message. On the other hand the first "H" in the coded message corresponds to a "–" in the original while the second "H" corresponds to an "H" in the original.

The receiver now opens his vault and calculates the inverse of the matrix B by our theorem on inverses. Here $D=(-1)(1)-1\cdot 2=-3$ and we must find an *integer*, D^{-1}, such that $DD^{-1}=1$ modulo 29. (It

may be shown that such inverses exist for *all* non-zero elements of A_m if and only if m is a prime.) Now, there are algorisms for finding D^{-1}, but it suffices here simply to try the (finite!) number of possibilities. Thus, $(-3)\cdot 1 \neq 29$, $(-3)(-1)\neq 29, \cdots$, until, finally, we obtain $(-3)(-10)=+30 = 1$ modulo 29. Thus $D^{-1}=-10$ and hence

$$B^{-1}=\begin{pmatrix} (1)(-10) & -(1)(-10) \\ -(2)(-10) & (-1)(-10) \end{pmatrix}$$

$$=\begin{pmatrix} -10 & 10 \\ 20 & 10 \end{pmatrix}=\begin{pmatrix} -10 & 10 \\ -9 & 10 \end{pmatrix}.$$

Now the decoder writes down "S, J, $\cdots$ J, E", and the corresponding numbers according to our table, blocks these numbers off in groups of four, and forms the proper matrices which, of course, will be the $BA_1, BA_2, \cdots, BA_5$ previously listed. Now he multiplies each of these matrices by B^{-1} (*on the left*) and since $B^{-1}(BA_1)=A_1, \cdots, B^{-1}(BA_5)=A_5$ he will, of course (if our work and his is correctly done), arrive at our original matrices $A_1, \cdots, A_5$, from which it is easy to obtain the original message. For example,

$$B^{-1}(BA_1)$$

$$=\begin{pmatrix} -10 & 10 \\ -9 & 10 \end{pmatrix}\begin{pmatrix} 10 & 14 \\ -5 & 9 \end{pmatrix}=\begin{pmatrix} -150 & -50 \\ -140 & -36 \end{pmatrix}$$

$$=\begin{pmatrix} -5 & -21 \\ -24 & -7 \end{pmatrix}=\begin{pmatrix} -5 & 8 \\ 5 & -7 \end{pmatrix},$$

where we have divided 150 by 29 to get a remainder of 5, 50 by 29 to get a remainder of 21, and 140 by 29 to get a remainder of 24.

Clearly any other 2×2 matrix with $D\neq 0$ could be used as a key, and there are over 600,000 such distinct matrices. Or, we could agree ahead of time to use $A_1B, \cdots$ instead of $BA_1, \cdots$; one could use A_{31}, or, in general, A_p, for any small prime $p>31$. Thus the number of different codes based on the general principles just discussed is extremely large.

Have you read?

MERRILL, D. M. "Second Thoughts on Modernizing the Curriculum," *American Mathematical Monthly*, January 1960, pp. 76–78.

When a curriculum is to be studied and revised there are always many points of view. The present movement in curriculum revision is no exception. The author wonders if the heresy of yesterday is becoming the orthodoxy of today. Is the old curriculum obsolete, and if it is, will the new correct all the ills of the present? Much emphasis is placed on structure and language; at the same time, Euclidean geometry, the very course where language and structure is most evident, is being deleted. Does the new mathematics stimulate and build appreciation? Will the new be shaken down into a teachable form as dry as the old? Should we have a mathematics appreciation course? These and other questions you may want to consider. Read the article, but come to your own decisions.

You may want your students to read the article *A Tournament Problem* in the same issue. They will enjoy it.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

ROSENBERG, HERMAN. "Modern Applications of Exponential and Logarithmic Functions," *School Science and Mathematics*, February 1960, pp. 131–138.

As teachers we are always on the alert for ideas we can use in our everyday instruction in mathematics. This article contains just that kind of ideas. It presents the idea of function and its use in the sciences and social sciences. For example, the simple exponential function $y=ae^{kx}$ can be applied to the growth and decay of human beings, trees, money, wounds, electric currents, and many other phenomena. The logarithmic function $y=a \log_b u$ is useful in dealing with speed in a telegraphic cable, the pursuit of fighter planes, the demand for a commodity, the cost function of a factory, the eating of delicious foods, or seeing someone in love. The population curve is $y=e^x$, and the rate increases as the population increases, while, in love, $y=\log_e x$—the rate of increase slows down as love increases. As the Chinese Philosopher says, "Love is not logical—it is log$e$-ical."—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

*A new role for high school geometry**

ROBERT R. CHRISTIAN, *University of Illinois, Urbana, Illinois and The University of British Columbia, Vancouver, British Columbia.*

Through the study of geometry, students can meet some elementary examples of important abstract ideas.

As most of the readers of The Mathematics Teacher are well aware, mathematical education today is in a state of great ferment. A number of programs of reform are being carried out in various places on this continent, and many people are expressing their opinions as to what mathematical education should be, and what it should not be. In such a situation, it is natural for people to disagree. It is all the more significant, therefore, when people do agree.

One principle of mathematical education on which there seems to be widespread agreement is the following: The most effective way to learn mathematics is to proceed from the concrete to the abstract. Or, since all of mathematics is abstract, the most effective way to learn mathematics is to go from the less abstract to the more abstract.

The abstractness of mathematical ideas needs to be emphasized. As Morris Kline has pointed out in his book, *Mathematics and the Physical World* (page 23),

> "On the basis of such abstractions [from certain properties of physical objects] mathematics creates others that are even more remote for anything real. Negative numbers, equations involving unknowns, formulas, derivatives, integrals, and other concepts we shall encounter are *abstractions built upon abstractions.*" (Italics mine.)

Few people can grasp an abstract mathematical idea without working with examples of this idea. People need experience with elementary forms of an abstract idea before they can fully appreciate or comprehend the refined idea itself. I would like to suggest that high school geometry can play an important part in providing this experience.

It seems to me that there are two reasons why geometry is an ideal place to introduce elementary examples of abstract ideas. First, the subject has great intuitive appeal, and this helps to make examples easy to understand. It is possible, and natural, to draw diagrams to illustrate the material of geometry, and it is well known that a picture is worth a thousand words. Second, geometry is one portion of high school mathematics that is really profound. For this reason, the examples used need not be trivial.

As an illustration of my general remarks, consider the abstract idea of function. This idea is one of the few that seem to crop up everywhere in mathematics and science. You can find functions in physics, in biology, in geometry, in algebra, in calculus, and in virtually all branches of more advanced mathematics. Since the idea is important, it should be introduced as early as possible (in rudimentary form, of course), so that the student will have plenty of time to grow up with it.

To a specialist in the foundations of mathematics, functions are rather simple objects. Such a specialist may take the idea of function as primitive, or, with very little trouble, he may define functions in terms of sets of ordered pairs. In spite of the *logical* simplicity of this concept, however, students often have trouble

* This material was originally presented at a joint meeting of the National Council of Teachers of Mathematics and the Mathematical Association of America at Chicago, Illinois, on January 30, 1960.

understanding it. Now, why should this be so? Why do students have trouble with such a simple idea? The reason is not hard to find, when one considers the students' background. This background usually contains little that is even remotely connected with the idea of function. Thus, students often do not see what functions are for, and they are further confused by functional notation.

To understand and appreciate functions, students must be familiar with a number of examples of functions and their applications. And excellent examples can very easily be introduced into high school geometry.

Consider the case of a parallel projection of one line onto another. This is a special kind of function. Furthermore, it can be studied by itself, without any reference at all to the more general and abstract idea of function.

Think of two lines in a plane, and think of a third line that intersects each of these lines in a single point.

Call the first line 'l,' the second line 'm,' and let n be the line that intersects both.

Now think of the collection of *all* the lines parallel to n. Every line in this collection intersects both l and m. So, you can associate the point where one of these lines crosses m with the point where that line crosses l. Clearly, these associations give us a function whose domain is the line l and whose range is the line m. This function is an example of a parallel projection. [See Figure 1.]

In talking about parallel projections it is easy and natural to introduce functional notation. The parallel projection pictured in Figure 1, for example, associates U with A, V with B, etc. Denoting this parallel projection by 'p,' you can write '$p(A) = U$' to mean 'the point associated with A by the parallel projection p is the point U.' Again, in Figure 1 we have $p(B) = V$ and $p(C) = W$. Notation for the inverse of a parallel projection can be introduced, too. In the diagram, $D = p^{-1}(X)$ and $E = p^{-1}(Y)$. Note what this accomplishes. Without a lot of fuss and bother, this exemplifies the ideas of function and inverse of a function, and introduces notation for both these concepts.

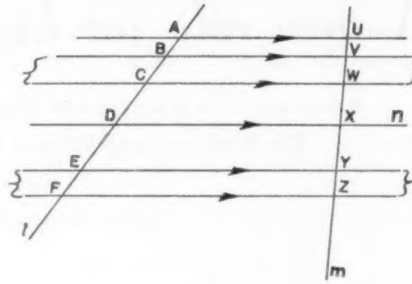


Figure 1

Parallel projections are not merely devices for introducing functions and functional notation. They have a deeper significance. To show that they do fit into geometry quite naturally, let us restate two of the standard theorems of high school geometry as theorems about parallel projections. First, consider the following theorem:

If three or more parallel lines intercept congruent segments on one transversal, then they intercept congruent segments on any transversal.

This theorem is really an easy consequence of a more powerful theorem on parallel projections, which reads as follows:

Parallel projections preserve congruence.

What this theorem says is illustrated in Figure 1. If the segments BC and EF are congruent to each other, then the corresponding segments VW and YZ are also congruent to each other. That is, the congruence of BC and EF is 'preserved' in the corresponding segments VW and YZ. This theorem can be proved with standard techniques of standard courses.

Note that the first theorem is an easy consequence of the second, but the second is not an easy consequence of the first. Thus the second theorem is mathematically the more powerful of the two. Since

the theorem on parallel projections is just as easy to understand and prove as the theorem on transversals, and since it is obviously easier to remember, it is evident that it is the theorem of choice.

Let us now reformulate a second theorem, the Basic Similarity Theorem. This theorem reads as follows:

If a line intersects two sides of a triangle in distinct points, and is parallel to the third side, then it cuts off segments which are proportional to these two sides.

Using the idea of a parallel projection, we can say the same thing—in fact, we can say a bit more—much more simply, as follows:

Parallel projections preserve ratios.

To see how the latter statement includes the former, examine Figure 2. In this figure, the points A, E, and C are associated with the points A, D, and B by means of a parallel projection of line AB onto line AC. The ratios $AD:AB$ and $DB:AB$ are preserved by this parallel projection. That is, $AD:AB=AE:AC$ and $DB:AB=EC:AC$.

The theorem that parallel projections preserve congruence is clearly a specialized version of the theorem that parallel projections preserve ratios. (In the first theorem the ratio of the given segments is 1.) Of course, both theorems should be studied. The fact that the second theorem is a generalization of the first, however, reduces the burden on the student's memory. So, of course, does the simple wording of both theorems.

Note that in none of this does one need to talk about the general notion of function, much less use the word 'function.' Nothing need be given but a *concrete example* of a function.

Another concrete example of a function is that of a *reflection* in a line or in a plane. This idea usually appears in discussions of symmetry about a line or a plane. Think of a line in a plane, and with each point in that plane associate its mirror image with respect to the line. This association is an example of a reflection.

If we denote the reflection shown in Figure 3 by 'r,' then $r(A)=U$ and $r(U)=A$. That is, the reflection of A is U and the reflection of U is A. Similarly, $r(B)=V$ and $r(V)=B$. Clearly, a reflection is its own inverse.

Reflections have very interesting properties. For example, we note that the reflecting line is its own reflection. In Figure 3, line l is its own reflection. Similarly, any line perpendicular to l is its own reflection in l. Further, l and the lines perpendicular to l are the only lines that are their own reflections. Reflections can easily be studied right along with perpendicular bisectors and angle bisectors.

In view of the fact that every segment (and every triangle, circle, etc.) is congruent to its own reflection, certain standard theorems become quite clear, intuitively, when viewed in terms of reflections. For example, experience with mirrors can make it intuitively clear that every point on the perpendicular bisector of a segment is equidistant from the ends of the segment, and every point on the bisector of an angle is equidistant from the sides

Figure 2

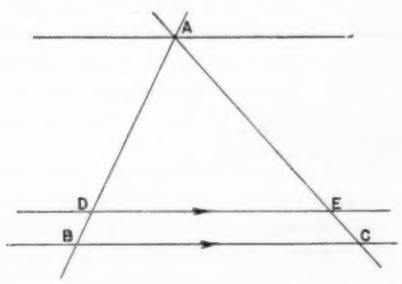
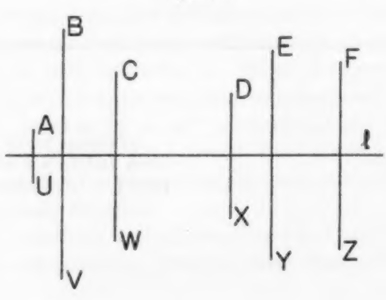


Figure 3



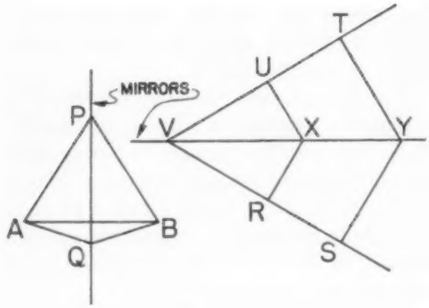


Figure 4

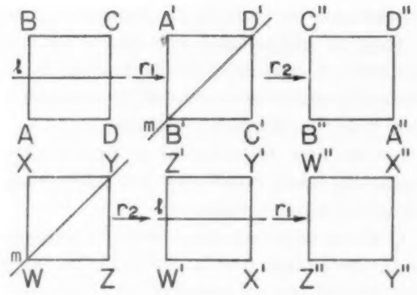


Figure 5

of the angle. [See Figure 4.] Intuitions of this sort are valuable as background for later mathematical study, even when not followed up by proof.

Work with reflections can easily lead to certain results not ordinarily found in high school courses. For example: Every rotation—which, by the way, is another kind of function—can be regarded as a composition of two reflections. If you first reflect in one line, and then reflect in another, intersecting, line, you effect a rotation. This is illustrated in Figure 5. In this figure, r_1 and r_2 are reflections in lines l and m, respectively. The diagrams in the first row show what happens when you first reflect in l, and then reflect in m. The diagrams in the second row show what happens when you first reflect in m, and then reflect in l.

In Figure 5, each reflection, r_1 and r_2, effects a rotation of 90°. The direction of the rotation depends on which reflection acts first. In the first row, we first apply r_1 and then r_2. The net result is a rotation of 90° counterclockwise. In the second row we apply r_2 and follow with r_1. This time the rotation is clockwise.

A more advanced result is that *any* 'rigid motion' is a composition of at most *three* reflections. I will not go into details.

In this paper I have presented the thesis that high school geometry can play an important role in introducing students to very abstract mathematical ideas. I believe that high school geometry can do this by providing relatively concrete examples of such ideas in action. I have illustrated this with the concept of function, although I could just as well have chosen other concepts for this purpose. I have attempted to show that certain concrete examples of functions can simplify the conception, and even the wording, of certain theorems in standard courses. Finally, I have tried to indicate that some concrete examples of abstract ideas are worthy of study in themselves, and form the basis of many interesting results.

"Persons looking for some particular flower often fail to see other flowers, no matter how pretty."—*F. Cajori.*

The SMSG geometry program

EDWIN E. MOISE, *Harvard University, Cambridge, Massachusetts.*

What should be done about geometry teaching in the high school program? The School Mathematics Study Group has some promising answers.

I THINK that anybody who cares about the progress of education ought to be encouraged by the activity that has been going on in the past few years. Lots of people seem to be excited, in a healthy sort of way, and lots of people are working, and thinking, and trying things. Of course they aren't all doing the same things, because they are trying to be original. And they don't always agree, because they are thinking for themselves.

Actually, I think that there are three main kinds of experimenters at work. There are the bold experimenters, like Max Beberman of Illinois; there are the curricular planners, such as the Commission on Mathematics of the College Entrance Examination Board; and there are the sample textbook writers in the School Mathematics Study Group. There are differences in viewpoint among these individuals and groups; and I think that these differences are due mainly to the fact that the three groups are tackling three rather different jobs.

The people that can work most freely, and most boldly, are the ones, like Max Beberman, who can try very original programs, on a scale small enough for safety. If you are working this way, then if something doesn't work, you simply try something else. If a program calls for very special teacher training, then you expect that the program will expand merely as fast as teachers can be trained to handle it. I have sometimes heard it claimed that the Illinois group is away out in left field, but if so, I think that we all ought to be thankful for left field, because that is where the real pioneering is done.

For us on the SMSG, the job is in one way easier and in another way harder. It is easier, because we aren't expected to make big leaps into the future. It is harder, because we aren't allowed to try. Our job, as we understand it, is to design a program that can be put into effect, in any high school that wants to use it, within the next few years. This sets limits on what we can try. For example, the experience of this year indicates that the average teacher can handle the SMSG geometry with an hour of inservice training a week. This is about what we were aiming for. Possibly in the future it can be reduced.

The Commission's job was in a way more restricted still. We all remember how and why its work was begun. The Board had realized that anybody who writes examination questions controls curricula; and if you are in fact controlling curricula, you ought to do it intelligently, instead of merely standing still and allowing yourself to become a roadblock. The Commission therefore got to work and drew up a plan that seemed to them to be the logical next step in the development of courses. Their program, like ours, was supposed to be usable by everybody, in the near future. And they had a further problem, which SMSG does not have: they drew up merely an outline, with illustrative text material in short fragments. Therefore their program was supposed to be written up in full, and carried out, by *other people*. This meant that their program had to be not merely feasible but obviously feasible.

Later on, I shall compare the Commission's geometry program with that of the

SMSG. The comparison will indicate that the SMSG program is far more ambitious. But in making the comparison, I don't mean to suggest even for a moment that the Commission was timid. The Commission was not timid. For one part of their program, in probability theory, they wrote a book; and anybody who has looked at their book is aware that the authors are extremely imaginative people. I suspect that the cautious character of their geometry program was due to the fact that instead of tackling the problems themselves, they were urging others.

When the SMSG geometry group met at Yale two years ago, we began by considering the Commission's recommendations. You will recall that the first and primary objection that the Commission made to synthetic geometry was that the logic of the treatment was not up to modern standards. In the last century, the level of rigor in modern mathematics rose past that of Euclid, and so modern mathematicians did the foundations of geometry all over again. In Hilbert's book you will find a *very* rigorous treatment. The Commission thought—and we in SMSG agree—that Hilbert is not the answer for the tenth grade. In response to these troubles, the Commission proposed, first, to reduce the number of synthetic theorems and, second, to avoid the problems in foundations by the use of analytic methods.

In our first discussions in SMSG, it was soon plain that nearly all of us had a different viewpoint from that of the Commission. In the first place, we were not convinced that the logical flaws in Euclid were fatal for the purposes of the tenth grade. It is true that Descartes invented analytic geometry, but he didn't invent it to patch up the foundations. He invented it to solve hard problems. Meanwhile, he accepted Euclid as the great model of rigorous deduction, and he tried to copy this model in his philosophical work. In a pinch, the kind of logic that served as a model for Descartes ought to be good enough for a fifteen-year-old to put up with, at least for a while. We on SMSG wanted to raise the level of rigor if we could, but we didn't consider that this was the central problem.

We were more seriously worried about rather different flaws in the traditional treatment. In the first place, it seemed to us that geometry was artifically isolated from the rest of mathematics. In Euclid's time, this was natural enough, because the rest of mathematics consisted entirely of the theory of numbers. The algebra of the real numbers simply didn't exist, except insofar as Euclid constructed it geometrically. Therefore Euclid had the job of axiomatizing literally everything in sight; there was nothing outside his system to which he could appeal to simplify his task. A couple of thousand years later, it has become quite unnatural to treat geometry in isolation from algebra. We were also concerned about the use of language in ways that seem odd in the context of modern mathematics. Finally, it seemed to us that this very ancient material could be presented in such a way as to be modern in an important sense. Perhaps the easiest way to sum up our conception of what ought to be done is to give three words that we think ought to describe a good treatment of geometry. The words are *integrated*, *rigorous*, and *modern*. All three of these words, I think , have been much abused lately, and perhaps I had better explain rather fully the meanings that we attach to them.

First, the meaning of *integrated*. There is a great deal of basic unity in mathematics. A genuinely integrated treatment is one which brings out this unity, in the cases where it really exists, and uses it to improve understanding. For example, algebra and geometry ought to be integrated. If you think of the real numbers as points on a line, then lots of things about the real numbers become easier to visualize and easier to understand. The same thing works in reverse: If you know about real numbers, then some things about the geometry of lines become easier to under-

stand. When you bring out these connections, you have a genuine integration; and this is what you get when you use the Birkhoff postulates for geometry.

In the Birkhoff treatment, the real numbers are supposed to be given at the outset; they are used to measure both distances and angles; and many of the postulates are stated in terms of the distances between pairs of points and the degree measures of angles. Later on, I will explain the advantages of the Birkhoff postulates from the standpoint of rigor, but that is not the sort of thing that I am driving at now. As a matter of fact, I have to admit that it would be misleading to say that in Birkhoff's treatment, the real numbers are assumed to be known. We all are aware that the real numbers are *not* known, with any degree of thoroughness, to students entering the tenth grade. The point is that in Birkhoff's treatment, the connection between lines and real numbers is made immediately. This means that geometry and algebra can immediately begin making their natural contributions to each other in the mind of the student. And it means that the gaps that we leave, in the logic of the geometry, are gaps of a sort that are going to be filled when the student learns more algebra. This integration is genuine.

Perhaps I can make this point clearer by giving an example of something which seems like an integration, but which in my opinion is an illusion. If you introduce co-ordinate systems in the plane, and prove elementary theorems by analytic methods it may seem that you are integrating synthetic and analytic geometry. There is no question that synthetic geometry makes a contribution to analytic geometry. As a matter of fact, you have to know quite a bit about synthetic geometry before you can set up a co-ordinate system, derive the distance formula, and find slopes of lines. But the contribution made in the reverse direction is terribly small. Elementary geometric theorems are usually easier to prove synthetically, and the synthetic proofs are prettier and more instructive. Only when you get to problems far beyond the tenth grade do co-ordinate systems become a good method of handling synthetic geometry. In the tenth grade, analytic geometry is not really a method at all. It is a brand-new subject in its own right. It ought to be introduced, I believe, but at the end of the course. It ought not be to put in the middle, because there it is merely a digression.

Another example of a genuine integration is one between plane and solid geometry. This, I think, was among the best of the Commission's proposals. There are lots of analogies and interrelations here that are worth bringing out. And the combined treatment has an extra advantage. If you start solid geometry early, then throughout the course you can use three-dimensional problems as exercise material. This means that you can give the student lots of intuitive experience with figures in space, without taking on the burden of a full deductive treatment. Personally, I think that this second feature is the more important one.

Let me now try to explain what I mean by *rigorous*, for the purposes of the tenth grade. Perhaps I should begin by describing the kind of rigor that I *don't* mean. Landau's *Foundations of Analysis* is as rigorous as any mathematical book that I know. Sad to tell, it is also the dullest mathematical book that I have seen; I have never managed to finish it, myself, and I have never expected a student to. I think most people agree that the endless dotting of i's and crossing of t's is not very inspiring and not very educational. In high-school teaching, rigor ought to take the form of disclosure, the form of candor. Rigor is saying what you really mean. I think it is a mistake to suppose that this always makes things harder for students.

For example, in our geometry, we use a symbolism that some of our own group thought was too complicated. Given two points A and B, there are four different things that we may want to talk about. These are the distance between A and B, the line that contains A and B, the seg-

ment from A to B, and the ray that starts at A and contains B. Some of us were afraid that the use of four different notations might be a burden to the student. But one day this year, one of us who was teaching a class using our book remarked to his students that, in a certain standardized test, the same simple notation AB was used to express all four ideas. The whole class groaned in unison. A large part of the time it seems to be a fact that when you say what you mean, this makes it easier for people to understand what you say.

The same principle applies, I think, to the use of postulates. The Birkhoff postulates, in the reworked form in which we give them, are really adequate to prove the theorems, without fudging. The treatment that we give to the student is not logically complete; some details were left out, as being too pedantic to insist on, and it may well be that we ought to leave out more things like this in the second edition. But the treatment that we give to the student is very close to being logically complete. We have tried to set up the thing in such a way that the rules of the game can really be followed, and so that the deductive process really is what it seems to be. It would be foolish to try to make a theoretical prediction on how well this works. The only way to find out how something works in the classroom is to try it. But my hope is that experience in genuine deductive reasoning will turn out to be more valuable, and also more fun, than experience in somewhat defective reasoning. I am not trying to claim that an all-or-none principle operates here, and I am not trying to claim that experience in the traditional treatment isn't valuable. I merely hope that we can do better.

I think that one of the finest things about the study of geometry is the opportunity that it offers for the student to be creative. One of the sad things about most mathematics courses is that the courses tend to break into two parts. First there is the so-called *theory* which is given in the text. Then there is the drill that the student goes through, in the form of endless exercises, mainly in the form of calculations. Every student knows that he is responsible for the drill; if he can manage to get the answers that are given in the back of the book, then everybody will have to admit that he has been a good boy. Sometimes we may tell the students that they "are not responsible for the theory"; when we do the sighs of relief can sometimes be heard down the hall.

It seems to me that this is a pattern that we ought to try to avoid in all of our courses. The special virtue of geometry is that in geometry the separation between the mathematical theory and the students' work is easy to avoid. We have been avoiding it for generations. The problem material in geometry is not a chore, or an elaborate form of social conformity. As soon as you get to the first congruence theorems, the students can start doing problems that use the imagination. The originals in a geometry course are the same sort of theorems that get proved in the text. This means that geometry is not a spectator sport; it brings the student right into the game. There is an old educational maxim that says that we learn by doing. I am not sure that this maxim has always been applied correctly, but I think that it does apply correctly to the present case. I think that the best way to learn what mathematics is and how it works is by doing mathematics. Synthetic geometry is one kind of mathematics that quite young students can and will do. I think that this feature gives synthetic geometry a very special kind of importance, even above and beyond the importance that the subject matter has in itself.

Let me now explain the sense in which I think that ancient mathematics ought to be made modern. The idea here is not simple. Just as there is true integration and false integration, and true rigor and false rigor, so there is true modernness and false modernness.

Mathematics is a very peculiar science: it is the only one that has no funerals.

Most of the scientific theories of Plato's time are of no scientific interest today, for the simple reason that they were wrong. The mathematics of Plato's time is not only still right, but still good. But this does not mean that mathematics simply accumulates geologically in successive strata. Very often, when we learn something new, this throws a new light on what we knew already. The old mathematics may still be good, and may still be important, but the way in which it was formulated may begin to seem unnatural. The face of calculus has now changed to a point where its own father wouldn't know it. And modern mathematicians would have great trouble in reading the work of Sir Isaac Newton, even if they tried, which they don't.

Thus a modern treatment of old mathematics is a treatment that fits the old mathematics into the total picture that exists today. Sometimes, the sort of changes that make a treatment modern look exciting and spectacular. But often they involve merely a long string of quiet changes in choice of language and matters of detail. Let me give one simple example.

In the last century, the idea of a set has become one of the central ideas of mathematics. It comes up everywhere, and it always means the same thing. But this does not mean that the idea of a set is new. Sets have been known, at least in some sense, since the Stone Age. It was in the Stone Age that men learned to count. If you know how to count, then you assign the same number five to a group of five men and a herd of five mastodons. When you use the same number in each case, you are not describing a common property of men and mastodons. What you are describing is a common property of two sets. Thus the idea of a set was implicit in Stone-Age mathematics. It seems likely, of course, that the set theory of the Stone Age was rather primitive. And only very recently did the word *set* come to be used whenever the idea of a set was intended.

In a modern treatment of any mathematics at all you ought to use the word whenever you use the idea, so as to bring out the genuine unity that is involved. For example, the circle with center P and radius r is the *set* of all points whose distance from P is equal to r. The perpendicular bisector of a segment is the *set* of all points that are equidistant from the end-points of the segment. The graph of an equation, in analytic geometry, is the *set* of all points whose co-ordinates satisfy the equation. And so on and on. This may not seem very exciting. You may remember that I never claimed that it was. But it will make quite a difference to the student. He will find the abstract concepts of modern mathematics far easier to understand if he has seen them applied, over and over again, in concrete situations that he understands completely.

This is an example of what I mean by true modernness. Sets will also do very nicely as an example of false modernness; it all depends on how you talk about them and when. There is very little point in treating the so-called theory of sets as a separate item in the curriculum. The point is that the theory of sets breaks sharply into two parts. First, there is the trivial part, which amounts to hardly more than a choice of language. The facts about sets that you use in an elementary course are very simple facts, which everybody knows as soon as he understands what the words mean. Then there is the advanced part of set theory, which is one of the hardest branches of modern mathematics.

One way to introduce sets into high school study is to treat them as a separate topic and to go on using the old language in the rest of the mathematics. This, I think, is false modernness. There are two things wrong with it. In the first place, very elementary set theory is hardly a theory at all; it is hardly more than a language, and to try to inflate it into a theory is misleading. In the second place, there is no point in introducing a new language unless you are going to use it.

You may remember that Tom Sawyer's Aunt Polly wore glasses. Sometimes she

looked under them, and sometimes she looked over them, but she seldom if ever looked through them—she was wearing them just for show. This is not the sort of example that we ought to be following.

This is the best short explanation that I can give of the philosophy behind the SMSG geometry. The first year of experimental teaching is over, and when all of the results are in, we ought to have a realistic notion of how well it works. The testing is severe, as it should be. It is well known that educational experiments usually work if you limit the teaching to a small group of highly-trained converts. As a matter of fact, I have hardly heard of a case in which an author has taught his own text without reporting success. In the SMSG experiments, this strategy is not used. Every text is being tried in about seventy-five classrooms. Far from being converts, nearly all of the teachers were taking a pig in a poke: they hadn't even seen the texts until a few days before the start of the term. The scattered reports that I have heard so far are reassuring; they indicate, at least, that we were not far off base, and in particular that the course is of about the right length. For a more exact idea, we will have to wait until all of the returns are in.

Finally, perhaps I should explain what the differences are, as I understand them, between the SMSG text and the Commission's program. The Commission's outline for the tenth grade covers pages 38 and 39 of their report. All of the topics listed are covered in our text, with small and scattered exceptions given in a few lines of the outline. With these exceptions, we include all of the topics that the Commission recommended, and in particular all of the analytic geometry that they wanted.

On the other hand, we include a great deal of material that the Commission did not mention. Some of this extra material is as follows: First, a chapter on geometric inequalities. Second, a derivation of the area formula for a circle. Third, a section on solid mensuration; here we give derivations of the usual volume formulas. We did not agree that these derivations should be postponed to a calculus course. If they can be learned earlier—and we think they can be—then they form a valuable introduction to ideas that will later appear in much more complicated forms.

But this difference in the sheer amount of material covered is not by any means the only difference that is involved. In all of its main features, the Commission's program is surprisingly conventional. It is true that they undertake to integrate synthetic and analytic geometry; they propose to treat solid geometry intuitively; and they propose various innovations and improvements in matters of detail. Nevertheless, we can summarize the situation fairly well by saying that the Commission's tenth-grade program simply fits together truncated versions of existing conventional courses. In particular, the basic approach to plane synthetic geometry is the old approach, retaining whatever flaws the old approach may have. Only the duration of these flaws is reduced when the course is shortened. And the analytic geometry comes much too late to be of any help in the problem of the foundations.

In other words, at nearly every point where the two programs differ, the difference consists in the fact that the SMSG program is more ambitious. Perhaps we were too ambitious. If so, we will be called to order by the experimental results. But if we had been very far out of line, I am sure that by now we would know it. This means that with the possible exception of the intuitive solid geometry, the Commission's program is surely feasible. I don't see how it could fail to work, if the SMSG program comes anywhere near to working. As a matter of fact, I think that the Commission's program represents the uttermost limit of what we might have to retreat to. I think that most of the main questions have now been reduced to practical questions in the classroom, and I can hardly wait for the returns to come in. My fingers are crossed.

Determinants whose values are zero

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How can students develop imagination in mathematics unless they get opportunities to exercise their imagination like the one described here?

IN A COLLEGE freshman mathematics course, I wrote the following third order determinant on the board for the students to evaluate:

$$\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}.$$

I thought that I had chosen a nonzero determinant, but much to my surprise the majority of the students said that the value was zero. Expecting that the majority had simply made a common mistake (which sometimes happens) I evaluated the determinant and found that the value was indeed zero. Then, one of the better students asked, "Do you think that this determinant is also zero?"

$$\begin{vmatrix} 10 & 13 & 16 \\ 11 & 14 & 17 \\ 12 & 15 & 18 \end{vmatrix}$$

"I don't know," I replied, "but let's find out." The class then proceeded to evaluate this new determinant and found that its value also was zero. By now imaginative minds were creating all sorts of questions.

"What about this determinant?" another student asked.

$$\begin{vmatrix} 2 & 8 & 14 \\ 4 & 10 & 16 \\ 6 & 12 & 18 \end{vmatrix}$$

"Or this?" still another student inquired.

$$\begin{vmatrix} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{vmatrix}$$

Each of these determinants was evaluated and their values were indeed zero. (The fourth order determinant was evaluated co-operatively—one group of students evaluating one minor, another group another minor, and so on.) On succeeding days we developed several elementary properties of determinants, and the students were then able to see why the value of each of the determinants that we had first considered was zero.

The general problem is an intriguing one. If we consider a determinant whose order is greater than two and whose elements are real numbers, such that the elements of each row (column) taken in order form an arithmetic sequence, its value will always be zero. Thus if n represents an integer greater than 2, and a_1, a_2, a_3, $\cdots$, a_n and c_1, c_2, c_3, $\cdots$, c_n are real numbers, then

$$\begin{vmatrix} a_1 & a_2 & \cdots a_n \\ a_1+c_1 & a_2+c_2 & \cdots a_n+c_n \\ a_1+2c_1 & a_2+2c_2 & \cdots \quad . \\ a_1+3c_1 & a_2+3c_2 & \vdots \\ \vdots & \vdots & \vdots \\ a_1+(n-1)c_1 & a_2+(n-1)c_2 & \cdots a_n+(n-1)c_n \end{vmatrix} = 0.$$

By way of illustration, each of the following determinants will have value zero:

$$\begin{vmatrix} 1 & -2 & 3 \\ 3 & 2 & 6 \\ 5 & 6 & 9 \end{vmatrix} \qquad \begin{vmatrix} 1 & 5 & 9 & 13 \\ 13 & 9 & 5 & 1 \\ 4 & 6 & 8 & 10 \\ -1 & -5 & -9 & -13 \end{vmatrix}.$$

To prove that any determinant of order greater than two whose elements in each row (column) are in arithmetic sequence has value zero, we need only show that by the application of elementary properties of determinants each element of some row (column) can be replaced by zero.* It then follows that the value of the determinant is zero. Consider the elements of the kth column (k a positive integer) of the general determinant written above. The elements of the kth column that are in the rth, $(r+1)$st, and $(r+2)$nd rows (r a positive integer) are thus

$$[a_k+(r-1)c_k],\ [a_k+rc_k],\ \text{and}\ [a_k+(r+1)c_k].$$

Then,

$$(1)[a_k+(r-1)c_k]+(-2)[a_k+rc_k] +(1)[a_k+(r+1)c_k]=0.$$

Thus each element of the $(r+2)$nd row can be made zero, and hence the value of the determinant is zero. Again, by way of illustration, let us examine the determinant that appears in the next column.

* We could also show that each element of any row (column) may be expressed as a linear combination of the corresponding elements of any other two rows (columns).

$$\begin{vmatrix} 1 & 6 & -4 \\ 3 & 8 & -6 \\ 5 & 10 & -8 \end{vmatrix}.$$

If we multiply each element of the first row by 1 and each element of the second row by -2, and then add the results to the corresponding elements of the third row we have

$$\begin{vmatrix} 1 & 6 & -4 \\ 3 & 8 & -6 \\ 5 & 10 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 6 & -4 \\ 3 & 8 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 0.$$

Since the proof of our general statement utilized the elements of only three rows (columns), it seems clear that it is necessary that only three corresponding elements of each row (column) be in arithmetic sequence. Hence, if the corresponding elements of any three rows (columns), taken in order, of a determinant of order greater than two whose elements are real numbers form arithmetic sequences, the value of the determinant is zero. In the following determinant the elements in each column of the first, third, and fifth rows form arithmetic sequences.

$$\begin{vmatrix} 1 & -3 & 13 & 11 & -2 \\ 2 & 46 & 97 & 32 & -8 \\ 5 & 3 & 14 & 9 & 1 \\ 4 & 11 & -1 & 18 & 103 \\ 9 & 9 & 15 & 7 & 4 \end{vmatrix}$$

Thus, the value of this fifth order determinant is zero.

I am certain that students and their teachers will enjoy discovering determinants whose values are zero, as well as proving that their values *must* be zero.

"The young student of mathematics is apt to look on this subject as one of solving problems, but the more he advances the more truth he will see in the statement that mathematics is a science which enables one to escape the trouble of solving special problems."—*G. A. Miller*

NDEA support for improved mathematics instruction

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The National Defense Education Act offers financial assistance for the purchase of teaching aids and the employment of state supervisors of mathematics.

WE AS mathematics teachers must increase our knowledge and learn to use the tools that will make our work more effective. We cannot continue to teach our mathematics with a sheaf of notes, a piece of chalk, and a blackboard. We must teach differently and more effectively. Let us take a look at the progress made in several other fields.

How long would it take you to multiply 4,783,941,592 by 8,641,882,375? You might do it in five minutes, with an additional five minutes or so to check your answer. With a small, gear-operated adding machine one can do the job in twenty seconds. On a large-scale computer that uses telephone-type relays your problem can be worked in perhaps four seconds. But this is still a snail's pace compared to the use of more recent electronic techniques. It is now possible to multiply more than ten thousand pairs of ten-digit numbers in a single second.

Research has been under way on even more impressive devices. One computation laboratory is building an automatic dictionary which might perhaps be thought of as an elementary translating machine. It will be able to store about five thousand Russian words and their English equivalents. All words are expressed in binary digits, letter by letter.[1] Pfeiffer tells us that "a-n-d" would be "00011-01100-00100."[2] This may seem like a lot of numbers to represent a small word, but remember the computer can run through thousands of words before you can take a deep breath.

Fifty years ago the farmer was using the walking cultivator and the sulky plow. Now almost every job on the farm is mechanized. In the last twenty years, farmers have changed their methods more radically than in the previous two centuries. "Since 1945, they [farmers] have increased their number of newer work-saving machinery by 1,200%—mostly with machines that had not been invented in 1938."[3] The next big step for the farmer will probably be from mechanization to automation in the raising of animals and fowl. Some farmers today are copying the assembly line techniques of industry.

With the tremendous advances in all fields, it is probable that it would now require more time than there is in the day just to keep up with the progress of human knowledge. So it would seem that we must find new means of communicating mathematical information by techniques more efficient than we mathematics teachers have used in the past.

[1] John Pfeiffer, *The Human Brain* (New York: Harper and Brothers, 1955) p 247.

[2] *Ibid.*, p. 249.

[3] "The Pushbutton Cornucopia," *Time*, LXXIII, 10 (March 9, 1959), pp. 74–75.

The square on the hypotenuse of a right triangle equals the sum of the squares on the two sides. If you are like most teachers, you have taken several days in a ninth-grade algebra class to *tell* your students about this relationship. You review this theorem a number of times during the semester. Then finally you check at the end of the year as to whether the student has retained this information. To your dismay, you discover that he does not understand the theorem. Why has instruction failed to produce the desired changes? The answer is that we have placed too much emphasis on one means of conveying information or ideas . . . words. We *tell* our students. Words *are* marvelous, but if the technique is abused, words can produce serious trouble, chiefly "verbalism" and "forgetting." But place a teaching aid illustrating the Pythagorean Theorem at the student's disposal as you discuss the relationship, and you give him an opportunity to enter actively into the learning situation. Visual, auditory, tactual, oral, and muscular sensations join in a force that has tremendous influence in forming new and lasting thought patterns that aid in the retention of knowledge.

Multisensory devices could render effective assistance in overcoming some of the difficulties in mathematics of high school pupils and their teachers. These devices now may be purchased with the help of the Federal government through the National Defense Education Act.

The National Defense Education Act of 1958 authorizes something over one billion dollars in federal aid over a four year period. In the inclusive sweep of its ten titles, NDEA touches every level of education, both public and private.

Loans to college students available through Title II enable needy students to complete their higher education in numerous fields. Money for counseling and guidance is provided in Title V and should benefit all fields. This is also true for Title VIII which provides for research and experimentation in new educational media. Title III includes science and mathematics because there have been critical shortages of scientists, engineers, teachers, and technicians in these areas. The nation's fate, even in the immediate future, might hinge upon the extent to which we were able to meet the challenge of the competitive technologies in the Communist countries.

Financial assistance for strengthening science, mathematics, and modern foreign language instruction is offered through Title III of the NDEA. Under this Title, Congress authorized seventy million dollars each year for four years for science, mathematics and foreign language equipment and minor remodeling of laboratory or other space that would make such equipment more usable. Congress did not appropriate the maximum amount authorized during the first two fiscal years for this phase of the program. The actual appropriations for acquisitions were \$56,000,000 and \$60,000,000 in 1959 and 1960 respectively.[4]

Congress also authorized, through Title III, five million dollars a year for four years to be paid to State educational agencies for expanding their "supervisory and related" services to public elementary and secondary schools—in science, mathematics and modern foreign languages. The actual appropriations were \$1,350,000 and \$4,000,000 in 1959 and 1960 respectively. The states were not required to match the federal money in the first year for this phase of the program. This provision allowed states to organize and staff their departments even though state funds were not available during the first several months of operation. Dollar-for-dollar matching was required after the first year to carry on the supervision and related services of the program.

Before the NDEA there were only four states with specialists in mathematics and eight with specialists in science in state departments of education in the entire

[4] Twelve per cent of these sums was set aside for loans to private schools' programs.

United States.[5] As of January 25, 1960, 31 states and territories had science supervisors, 25 states and territories had mathematics supervisors, and 14 states and territories had one or more persons acting as combination mathematics and science supervisor. Many state departments would be glad to employ specialists in science and mathematics if more qualified persons were available. The increase in the number of specialists suggests the probable influence of the NDEA.

To assure more efficient use of federal funds and thus improve the quality of education, the Act calls for responsible action at every level. With the objectives of the Act in mind, the officials in state departments of education must develop or improve standards and give leadership and supervision to administrators and teachers in the local schools. You, as a mathematics teacher, will want to avail yourself of services provided by your state department of education.

Through the NDEA, a singular opportunity is presented to the mathematics teacher. The ingenious instructor, through the use of charts, diagrams, models, films, motion pictures, or other multisensory aids, will strengthen his mathematics instruction. Such aids have been used as a teaching device for some time. Even Euclid saw the value of aids of this sort. "The recognition of this need of models is found in the first English translation of Euclid's *Elements*."[6] One of the most interesting features of Euclid's book is that many of the figures are made of paper and are so pasted in the book that they may be opened up to make actual models of the space figures.

Today, rulers, compasses, T-squares, protractors, flexible wooden triangles and other polygons, spheres, cubes, sundials, sextants, and flannel boards are but a handful of literally hundreds of devices available to the enterprising teacher for increasing his communicative efficiency in plane and solid geometry. By the use of visual aids and equipment the imaginative teacher can accomplish a great deal. However, as the student progresses he should become less dependent on the concrete and develop an understanding of the abstract or purely mathematical.

Ask yourself as a mathematics teacher: What does the NDEA mean to me? Is the need for teaching aids in mathematics recognized in my school? What can I do to improve the instruction in mathematics? Cooperate with your local administrators and your state department of education to get all possible support for your mathematics program.

[5] "States Lacking Science and Math Supervisors," *Hearings Before the Committee on Labor and Public Welfare, United States Senate, Eighty-Fifth Congress, Second Session on Science and Education for National Defense* (U. S. Government Printing Office, Washington, D. C., 1958), p. 262.

[6] Lao Genevra Simons, "Historical Material on Models and Other Teaching Aids in Mathematics," in *Multi-Sensory Aids in the Teaching of Mathematics*, Eighteenth Yearbook of the National Council of Teachers of Mathematics (New York: Bureau of Publications, Teachers College, Columbia University, 1945), pp. 253–254.

"For scholars and layman alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?"—*Richard Courant and Herbert Robbins*, "What Is Mathematics?"

The role of the state supervisor of mathematics

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What does a state supervisor of mathematics do?

As I NEED hardly tell you, one of the important clauses of Title III of the National Defense Education Act provides for the establishment or expansion of supervisory services in each state. As a direct result of this, specialists in mathematics (having somewhat varying titles) have been appointed in a considerable number of state education agencies, and it is probable that almost all states will have employed people in such positions within a year.

Partly to assist such appointees, the National Council of Teachers of Mathematics has published a pamphlet titled *The Supervisor of Mathematics: His Role in the Improvement of Mathematics Instruction.* This paper, prepared by a committee of four under the chairmanship of Veryl Schult, contains an excellent summary of the desirable qualifications of a mathematics supervisor. It also outlines the responsibilities of such supervisors and suggests ways in which these responsibilities may be met. I recommend the booklet and endorse it generally.

Since the educational histories and laws of the different states vary considerably, not all of the activities and responsibilities outlined in the Council's pamphlet apply to every state. For instance, several items referring to the selection of textbooks for state adoption or for recommendation to the schools certainly do not apply to New York (where there are no such adoptions). The wise supervisor will use this booklet as a general guide and not feel that he must engage in every type of project mentioned.

In one sense the NDEA found New York in a somewhat unfortunate position, since we were already performing many of the services specified in the law before it was passed and are, therefore, unable to "establish" them, although, of course, we have been able to "expand" in these areas. For instance, there have been supervisors of mathematics in New York State for decades, and three such were on the staff when the world's altitude record was first held by a dog. New York has a history of relatively strong state control of education, with official curriculums and syllabuses backed up by uniform statewide end-of-course examinations (known as the Regents examinations) based on these courses of study. Subject specialists in each of the major academic areas have been a part of this system for almost a century. I do not mean to argue the merits of the system at this time, but I feel that it is necessary (lest you get a wrong impression) to point out that the system is not entirely inflexible—that experimentation is continuously encouraged and sponsored by the state, and that every major experiment in mathematics education in America today is being tried out in a few schools in New York State with the full approval and active assistance of the State Education Department.

Thus, in at least several ways, New York State is atypical, and I am an atypical supervisor of mathematics if for no other reason than the fact that 50 per cent of my salary is not paid from federal funds. Some of the things I will say probably do not apply in your state. All I can do here is tell you of some of the things we have done, now do, and/or plan to do. You

are at liberty to pick from this enumeration anything you think to be of value in your situation.

I consider the role of my unit to be fundamentally catalytic (in the sense of the *first* half of the usual definition but not of the second). My staff and I try to do all we can to improve mathematics instruction, to speed up reactions we feel are desirable and, much less frequently, slow down or even stop completely some that we know are not. In these times, considerable effort is necessary just to "stay on top of what is going on" in our field—to know the details of the various experimental programs and proposals in mathematics and to encourage active but limited participation by some of our schools in each of the more important of these programs.

It is absolutely necessary that the state supervisory unit in mathematics have an overall strategy. General goals, formal or informal, should be set up. These are subject to amendment and they need not even be written but they should exist. We are not sure that our goals can ever be attained but we seek to make the absolute value of the difference between them and the actual situation less than epsilon for t (time) sufficiently large. In our own case, we try to keep the following in mind.

1) Every child in the state who is mentally capable should be required to gain a reasonable minimum competency in mathematics with particular emphasis on arithmetic.

2) Every child in the state should have the opportunity, under competent instruction, to pursue the study of mathematics in the public schools to the level necessary for success in any program in any institution of higher education.

3) Every child in the state having unusual talent in mathematics should have an opportunity to progress in this subject at a rate commensurate with his abilities.

4) The level of mathematical education must improve, particularly in the area of fundamental mathematical thought, and this must be achieved without any reduction in the development of necessary manipulative skills.

Now, strategy is worthless unless it is backed up with good tactics. The following is a sort of annotated list of our principal tactical activities.

I. We make supervisory visits to individual schools. We feel these are valuable to the teachers of the school visited and are also of great value to us in our work, for through such visits we gain a first-hand knowledge of the situation in mathematics education in schools across the state. This helps us to fit our services to the widely varying conditions which exist.

II. We organize, attend and participate in conferences. It is largely by such conferences that we are able to provide leadership in co-ordinating the activities of teachers throughout the the state for the improvement of mathematical instruction.

III. With the assistance of committees, we do extensive work in the area of curriculum planning and study. We also issue numerous publications in the curriculum field. These publications fall roughly into three types:
   a) The official state syllabus which is mainly a statement of *what* should be taught.
   b) Teachers handbooks and resource units of a non-mandated nature which deal largely with *how* topics may be taught.
   c) Outlines of new or experimental courses for limited experimental use.

IV. In co-operation with the Bureau of Examinations and Testing, we supervise the committees of teachers composing state examinations in mathematics and rate these examinations.

These include the Regents examinations, scholarship examinations, progress tests, survey tests and special examinations. Some idea of the scope of these programs is indicated by the fact that more

than 225,000 Regents examinations in mathematics will be given in our schools this year.

V. We work with our division of communications to utilize the most modern media for instruction.

One of the ways in which top quality instruction can be made available in remote areas is by television (open and closed) and by kinescope recordings of such broadcasts. At the moment over 300 hours of 16 mm. sound films in mathematics, produced within the last two years as byproducts of television presentations, are available free of charge to schools and teachers in the state.

VI. We promote in-service training of mathematics teachers. We have a number of projects currently operating in this area.

Perhaps the most successful is a complete course called Mathematics for Teachers taught by Harry Ruderman of Hunter College High School and recorded on 89 half-hour reels of film, which are being shown to groups of teachers in 12 widely scattered centers. The films are also being broadcast on television in Rochester, and the programs were previously televised in New York and Albany. We are able to furnish these films to other states at cost, which is $40 per half hour reel.

We assist the division of higher education in setting up special courses for mathematics teachers in colleges throughout the state. The cost of these is covered entirely from money appropriated by our state legislature. We also assist faculties of some colleges in planning N.S.F. institutes.

VII. We assist in administering the N.D.E.A. in the state.

VIII. We assist in choosing those experimental programs in the schools which will be given special state assistance.

IX. We issue periodic supervisory letters.

X. We furnish "expert advice" on matters mathematical to the entire department. We also represent the department in its dealings with mathematicians and mathematical organizations.

XI. We make surveys. At the moment we are working with the Bureau of Statistical Services on one concerning enrollment in various types of courses.

XII. We work in a close and harmonious manner with persons in similar positions in neighboring states.

XIII. We work closely with the officers and special committees of the State Mathematics Teachers Association on mutual problems. They make their wishes known to the department through us.

The above listing is far from complete. Much of our tactics is conditioned by targets of opportunity and cannot be anticipated. We constantly answer letters and telephone calls from teachers varying from "Where can I see a teaching machine in action teaching mathematics?" through "Is there a school within 75 miles of mine which has a particularly excellent seventh-grade mathematics program that would be worth visiting?" to "Are the altitudes of a tetrahedron concurrent?"

If there is a simple way to sum it all up it may be this: The state supervisor of mathematics should have a knowledge of mathematics which will be respected by any teacher in the state and by faculty members of the colleges which are training teachers; he should be a teacher of recognized ability and with substantial experience; he should know what is going on in mathematics education in his state and in the nation; and he should use his knowledge and abilities in any way he can for the improvement of instruction.

The unusual success of the USSR in certain areas has made mathematics good politics. The state supervisors of mathematics have a central role in making that good politics into good education.

*Calculus in the high school**

W. EUGENE FERGUSON, *Newton High School, Newtonville, Massachusetts.*

Calculus is a proper high school subject, if taught by qualified teachers to adequately prepared students.

FROM 1940 TO 1954 I taught calculus to freshmen and sophomores in three different colleges and universities. In two of these, it took us a year to get most of the students ready for calculus in the sophomore year, and in the other, calculus and analytic geometry was the beginning freshman course.

At various times during this period I was asked by high school teachers, "In the senior year, after we have finished solid geometry and trigonometry, should we start teaching the calculus?"

In the brief moment after the question was asked all I could see was that miserable set of papers from my differential calculus class that showed that they really were weak in algebraic manipulation, so my answer became a standard one:

"No, don't teach calculus, teach them more algebra and how to carry through an algebraic manipulation involving complex fractions and fractional exponents without making errors."

Today, 1960, a short twenty years later, the idea of *not* teaching calculus in my high school would be unthinkable. Calculus *is* a proper high school subject. This does not mean that *all* high schools should offer calculus in their high school next fall! Some may never offer calculus, but many more should than do now.

Before a high school offers a calculus course in its mathematics department certain conditions *must* be met, by the *school*, the *teachers*, and the *students*.

1. The school must have a curriculum offering in mathematics that allows the student to complete the mathematical prerequisites for calculus by the end of the junior year. In my school this means that he has had elementary, intermediate and some college algebra, plane and some solid geometry, trigonometry and some coordinate geometry by the end of the junior year.

2. There must be at least one teacher on the staff who can teach a bona fide college calculus course on the college level as outlined in the Advanced Placement Program of the College Entrance Examination Board. This does not mean that a school must be in the Advanced Placement Program in order to teach a full year of analytic geometry and calculus in high school. Many high school teachers have gone back to school and renewed their acquaintance with calculus and are now doing an excellent job teaching the subject.

3. The student must be adequately prepared, mathematically, for the course and willing to spend 8 to 10 hours a week on homework. He must be adequately motivated for taking the calculus.

Note carefully that the type of calculus course I want in high school consists of analytic geometry and both differential and integral calculus. It is clear by now that I want calculus as a high school subject, and the type of course I am talking about is also clear. But why?

Some high school students can do calculus and enjoy it. They may be going into

* Talk given as part of a panel discussion at a joint meeting of the Mathematical Association of America and the National Council of Teachers of Mathematics on Jan. 30, 1960, in Chicago, Illinois.

the scientific field; calculus is a necessary tool in many college physics courses taught in high school. Some may be interested in other fields; a calculus course of this type is not a bad terminal course in mathematics. Some of these students will surely become mathematicians.

We have sorely underestimated the mathematical powers of children in grades K–12. For students who are bright enough (and there are more of them than we think) these new mathematics programs are getting rid of wasted time in the seventh and eighth grades as well as in other years. The programs also are raising the level of mathematical competence of many students in grades 7–11, so that at the end of grade 11 many students will be prepared for calculus.

Calculus in high school is successful. I would refer you to the successes of Harvard, Yale, Princeton and many other institutions with students who have taken calculus in high school. There are some failures, of course; that is to be expected; I had them when I taught in college, too. I know that some of you have had some sad experiences with students who have had calculus in high school, but most of those failures, I am sure, came from ill-conceived calculus courses, taught by ill-prepared teachers, taken by ill-prepared students. Many times this situation has been brought about by pressure from the public combined with poor planning by the administration and the mathematics teacher.

Specific cases are needed to bear out my remarks. These could come from many high schools in various parts of the United States, but you will pardon me for using my own school, Newton High School, the one I know best.

In the four school years 1955–59, Newton High School had 65 calculus students. On the Advanced Placement Examinations 20 per cent had high honors, 45 per cent had honors, 23 per cent creditable pass, 8 per cent pass and 4 per cent fail. I think this shows that we have been entirely too selective and have not allowed as many students to take calculus as we should. So we have 29 students in calculus this year and will have at least 50 next year. The number will continue to rise.

We gain the year for the calculus in two ways: give both intermediate algebra and plane geometry in the tenth year, or start algebra in grade 8 for the brightest mathematics students identifiable at the end of grade 7. I favor the last procedure.

Many people tend to blame student failure in first-year calculus in college on the fact that the student had a taste of calculus in high school. He is alleged to loaf the first few days, because he knows this material cold.

In our regular senior class of about 130 in mathematics, we give three weeks or so of work on polynomial calculus, culminating in the solution of maximum and minimum problems and curve tracing. Student feedback from the colleges indicates that this practice does not take the bloom off the freshman college calculus. The expressions "limit of a function" and "limit of a ratio" are greeted as friends, not strangers.

A first year of calculus in high school presents problems to some colleges, but these are good problems for them to have. Here are bright, well-qualified, eager students. The colleges that recognize this and make their programs flexible enough to take care of them are going to reap the rewards of admitting these students, who have great intellectual power.

We are now beginning to find new ways of developing the mathematical power of children beginning in the elementary schools; in a few short years many more students will have finished the regular four year high school program at the end of their junior year. At the present time calculus seems to be the proper course for them.

One last observation: When calculus is introduced into the high school mathematics program the quality of the other courses picks up. The teachers are stimu-

lated and so are students in other courses. And the calculus teacher does a better job in his other courses, too.

With all the work being done to improve mathematics programs in America, we should not neglect the many students who will soon be ready for the calculus in the senior year as a result of these very programs. Teachers trained and capable of teaching the calculus will see that these capable students are not neglected. There should be more summer institutes, academic year institutes and in-service training courses planned for potential calculus teachers. Let's get more teachers trained *now* to teach calculus tomorrow.

What's new?

BOOKS

Secondary

Analytic Geometry and an Introduction to Calculus, A. Clyde Schock and Bernard S. Warshaw. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Cloth, 165 pp., $3.96.

Business Mathematics, (2nd ed.), Exercises, Problems, and Tests, R. Robert Rosenberg. New York: McGraw-Hill Book Company Inc., 1959. Paper vi+218 pp.

Essentials of Business Arithmetic, (4th ed.), Edward M. Kanzer and William L. Schaaf. Boston: D. C. Heath and Company, 1960. Cloth, xiii+497 pp., $3.80.

High School Geometry, Rachel P. Keniston and Jean Tully. Boston: Ginn and Company, 1960. Cloth, iv+474 pp., $4.40.

Mathematics, First Course, John A. Brown, Bona Lunn Gordey, and Dorothy Seward. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Cloth, 323 pp., $3.64.

Mathematics, Second Course, John A. Brown, Bona Lunn Gordey, and Dorothy Seward. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960. Cloth, 365 pp., $3.40.

Solid Geometry, Royal A. Avery and William C. Stone. Boston: Allyn and Bacon, Inc., 1960. Cloth, viii+245 pp., $3.65.

College

Advanced Engineering Mathematics, (2nd ed.), C. R. Wylie, Jr. New York: McGraw-Hill Book Company, Inc., 1960. Cloth, xi+696 pp., $9.

Differential and Integral Calculus, James R. F. Kent. Boston: Houghton Mifflin Company, 1960. Cloth, xv+511 pp., $6.75.

Elementary Analysis, a Modern Approach, H. C. Trimble and Fred W. Lott, Jr. Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960 Cloth, xii+621 pp., $6.95.

Intermediate Algebra for Colleges, (2nd ed.), Joseph B. Rosenback, Edwin A. Whitman, Bruce E. Meserve, and Philip M. Whitman. Boston: Ginn and Company, 1960. Cloth, xii+315+xxxi pp., $5.

The Theory of Functions of a Real Variable, Volume One, James Pierpont. New York: Dover Publications, Inc., 1959. Paper, xii +560 pp., $2:45.

The Theory of Functions of a Real Variable, Volume Two, James Pierpont. New York: Dover Publications, Inc., 1959. Paper, xiii +645 pp., $2.45.

Miscellaneous

The Education of Teachers: Curriculum Programs, National Commission on Teacher Education and Professional Standards. Washington: National Education Association, 1959. Cloth, ix+464 pp., $3.50.

Mathematics Refresher, Kurt Walter. New York: Philosophical Library, 1959. Cloth, 96 pp., $3.75.

Praxis der Mathematik: Monatshefte der reinen und der angewandten Mathematik in Unterricht (Practice of Mathematics: Monthly of pure and of applied Mathematics in Instruction). Aulis-Verlag, Deubner and Company KG., Cologne, West Germany. 18 DM. per year plus postage.

Qualifications and Teaching Loads of Mathematics and Science Teachers in Maryland, New Jersey, and Virginia, Kenneth E. Brown and Ellsworth S. Obourn. U. S. Department of Health, Education, and Welfare, Office of Education, Circular No. 575. Paper, viii+101 pp.

Transactions of the Oklahoma Junior Academy of Science, Volume II, Oklahoma State University: Stillwater, Oklahoma, 1959. Paper, 117 pp., $1.

• EXPERIMENTAL PROGRAMS

Edited by Eugene D. Nichols, Florida State University, Tallahassee, Florida

At the present time many schools are experimenting with a variety of curriculum plans, teaching methods, and written materials. Beginning with this issue, THE MATHEMATICS TEACHER is planning to provide information on these many programs through this new department, "Experimental Programs."

If your school is engaged in an experimental project, you are invited to submit an article describing it to the editor of this department.

The objectives and activities of the School Mathematics Study Group

by John Wagner, Assistant to the Director, School Mathematics Study Group, Yale University, New Haven, Connecticut,

HISTORY

In the spring of 1958, after consulting with the presidents of the National Council of Teachers of Mathematics and the Mathematical Association of America, the president of the American Mathematical Society appointed a small committee of educators and university mathematicians to organize a School Mathematics Study Group whose objective would be the improvement of the teaching of mathematics in the schools. Professor E. G. Begle was appointed director of the Study Group, with headquarters at Yale University. In addition, the organizing committee appointed an Advisory Committee, consisting of college and university mathematicians, high school teachers of mathematics, experts in education, and representatives of science and technology, to work with the director.

The National Science Foundation, through a series of grants, has provided very substantial financial support for the work of the Study Group.

OBJECTIVES

The world of today demands more mathematical knowledge on the part of more people than the world of yesterday, and the world of tomorrow will make still greater demands. Our society leans more and more heavily on science and technology. The number of our citizens skilled in mathematics must be greatly increased, and understanding of the role of mathematics in our society is now a prerequisite for intelligent citizenship. Since no one can predict with certainty his future profession, much less foretell which mathematical skills will be required in the future by a given profession, it is important that mathematics be so taught that students will be able in later life to learn the new mathematical skills which the future will surely demand of many of them.

To achieve this objective in the teaching of school mathematics, three things are required. First, we need an improved curriculum which will offer students not only the basic mathematical skills but also a deeper understanding of the basic concepts and structure of mathematics. Second, mathematics programs must attract and train more of those students who are capable of studying mathematics with profit. Finally, all help possible must be provided for teachers who are preparing themselves to teach these challenging and interesting courses.

Each project undertaken by the School Mathematics Study Group is concerned with one or more of these three needs.

The School Mathematics Study Group is, of course, not the only organization concerned with the improvement of mathematics in our schools. Some of the others are the Secondary School Curriculum Committee of the National Council of Teachers of Mathematics; the University of Illinois Committee on School Mathematics; the Commission on Mathematics of the College Entrance Examination Board, and the University of Maryland Mathematics Project. Liaison with these groups is provided by the simple expedient of including representatives of these organizations on the Advisory Committee and in the work of the various panels.

Mathematics for junior high school

The School Mathematics Study Group believes it particularly important that greater substance and interest be given to the mathematics of grades 7 and 8. Our general point of view has been to think of grades 7 and 8 not as the end of elementary school mathematics, but rather as a foundation for the work of the senior high school. The curriculum for these grades should include a sound intuitive basis for algebra and geometry courses to follow.

To provide a concrete illustration of this kind of curriculum, textbooks for these two grades were prepared at a writing session at the University of Michigan in the summer of 1959. These texts were revised at a writing session at Stanford University in the summer of 1960. The revisions were based upon classroom evidence from the reports of about 100 teachers and 8,500 students.

Emphasized in the texts for junior high school are the following important ideas of junior high school mathematics: structure of arithmetic from an algebraic viewpoint; the real number system as a progressing development; and metric and nonmetric relations in geometry. These ideas are constantly associated with their applications. Time is given to the topics of measurement and elementary statistics. Careful attention is paid to the appreciation of abstract concepts, the role of definition, development of precise vocabulary and thought, experimentation, and proof. Materials are chosen with the intent to capture the fascinating features of mathematics, creation and discovery, rather than just utility alone. The texts provide adequate preparation for the remaining work in the SMSG sequence.

Accompanying each of these texts is a commentary for the teacher. These commentaries include not only the usual materials (discussion of teaching problems, etc.), but also discussion and deeper exposition of the mathematics.

Mathematics for high school

This project is devoted to the production of a series of sample textbooks for grades 9 through 12. We do not believe that any single curriculum is to be preferred to the exclusion of all others, and we intend the curriculum exemplified in these textbooks to be no more than a sample of the kind of curriculum which we hope to see in our schools.

For the most part the topics discussed in these textbooks do not differ markedly from those included in the present-day high school courses for these grades. However, the organization and presentation of these topics is different. Important mathematical skills and facts are stressed, but

equal attention is paid to the basic concepts and mathematical structures which give meaning to these skills and provide a logical framework for these facts.

We foresee a variety of ways in which these textbooks will be used. They will, we hope, provide a model and a source of suggestions for the authors of classroom textbooks of the future. A major use of these texts will be in connection with preservice and in-service training of teachers, since they will provide concerete illustrations of how an increased emphasis on basic concepts and on mathematical structure can be brought into the classroom. Finally, they will provide a stopgap till such texts become available through the usual channels.

The ninth-grade text, *First Course in Algebra,* emphasizes the structure of algebra. The study of algebra is based on the exploration of the behavior of numbers. Careful attention is paid to the language of the subject. While most of the material which traditionally makes up a first course in algebra is woven into the text, this material appears in an entirely different light since it is organized and motivated by fundamental structure considerations. This revised edition does not presuppose a study of the SMSG junior high school mathematics program.

Geometry is designed for the one-year introductory course in geometry which is usually taught in the tenth grade. This book is devoted mainly to plane geometry, with some chapters on solid geometry and an introduction to analytic geometry. Solid geometry is introduced very early in the text and is used as a foil to develop the students' space perception. The basic scheme in the postulates is that of George D. Birkhoff. It is assumed that the students are familiar with the number line, and the numbers are used freely for measuring both distances and angles. Geometry is connected with algebra at every reasonable opportunity.

Intermediate Mathematics, the eleventh-grade text, is written with the feeling that it is in the eleventh grade where able students reach higher ground in a mathematical sense. Careful attention has been paid to giving the student some insight into the nature of mathematical thought as well as to preparing him to perform certain manipulations with facility. Included in the text are chapters devoted to trigonometry, vectors, logarithms, mathematical induction, and complex numbers written with a much higher degree of sophistication than usually appears at this level. The text continues in the vein of the ninth-grade material emphasizing structure at all times.

Elementary Functions is designed for use in the first half of the twelfth grade, but by judicious use of supplementary material it could serve as a basis for a longer course. The central theme is a study of functions: polynomial, exponential, logarithmic and trigonometric, with emphasis on practical applications whereever possible. The introduction of a simple but geometrically meaningful method for handling areas, tangents, and maximum-minimum problems furnishes the student with a good intuitive background for a later course in calculus.

Introduction to Matrix Algebra is designed for the last half of the 12th grade. It is devoted to a study of matrices, including applications to solutions of systems of linear equations and to geometry. Careful attention is devoted to algebraic structure. Mathematics is introduced which is new to the student, and the structure is developed as the text proceeds. It is the intent of the text to put the student close to the frontiers of mathematics and to provide striking examples of patterns that arise in the most varied circumstances. A special set of "Research Exercises" is appended in the hope that some students may be introduced to real mathematical research.

Preliminary versions of these texts were prepared at a writing session held at the University of Colorado in the summer of 1959, using detailed outlines which had

been prepared at the Yale writing session in the summer of 1958. These texts were revised at a writing session at Stanford University in the summer of 1960. The revisions were based upon classroom evidence from the reports of about 350 teachers and 17,500 students.

As with the texts for grades 7 and 8, a commentary for the teacher accompanies each text.

Evaluation

A natural and frequently-raised question is concerned with the effectiveness of these texts both from the point of view of developing mathematical skills and from that of creating an understanding of concepts and structures. The latter will lead to greater problem-solving ability and longer retention of skills.

The use of the preliminary editions of the SMSG texts in experimental centers was concerned not so much with such an evaulation but rather with such questions as the amount of extra subject-matter-training teachers need to be able to use such new texts with confidence, the quality of the exposition, etc.

Can these texts be evaluated with respect to development of mathematical skills and problem-solving ability? Some statistical information, for the most part obtained from the Minnesota National Laboratory, and a great quantity of anecdotal material from the experimental centers are available. All information indicates that students using these texts do about as well in the development of mathematical skills, but do better in problem solving, than students using conventional texts.

However, more accurate information is needed, and a careful evaluation is now being carried out for SMSG by the Educational Testing Service. Preliminary results should be available in the summer of 1961.

Monographs

A third project is aimed at the production of a series of short expository monographs on various mathematical subjects. The primary objectives of such monographs are: to disseminate good mathematics at the secondary school level which will supplement the usual high school curriculum; to awaken interest among gifted students; and to present mathematics as a satisfying, meaningful human activity. The monographs are not intended as texts, but rather as supplementary reading material for students, their teachers, and the educated lay public.

Outstanding mathematicians are writing these monographs. In order to be sure that they are understandable and enjoyable to the audience for whom they are intended, preliminary versions will be read by high school students and experienced high school teachers. Their comments, criticisms, and suggestions will be passed on to the authors to form a basis for revision, if necessary.

These monographs will be published as paperbacks by a commercial publisher.

Teacher training materials

Practically all recommendations for improved secondary school mathematics curricula that have been seriously proposed, either by SMSG or by others, involve aspects of mathematics which have not, in the past, been included in the normal subject-matter training of secondary school teachers. Another SMSG project is devoted to the production of materials specifically for teachers who wish the additional training in mathematics needed to teach an improved curriculum. Particular attention is paid to materials suitable for use in summer and in-service institutes, such as those sponsored by the National Science Foundation.

Two series of publications are under way. The first is a series of study guides for teachers who wish to improve their professional competence by study either individually or in small groups. One of these, in algebra, is already available, and others on analysis, geometry, logic, number theory, and probability are in preparation.

The second series consists of brief expositions of various topics in mathematics designed explicitly for in-service teachers. Topics presently available in this series include set theory, geometry, and the mathematical background which teachers of the SMSG ninth and tenth grade courses will find useful. Others are planned for the other SMSG courses.

Mathematics for elementary school

In this project SMSG is undertaking a critical study of the elementary school mathematics curriculum from the point of view of increased emphasis on concepts and mathematical principles; the grade placement of topics in arithmetic; the introduction of new topics, particularly from geometry; and supplementary topics for the better students, for example, from number theory.

A start on this was made in the summer of 1960 in a writing session which prepared experimental materials for grades 4–6. These experimental materials are presently being tried out by about 200 teachers and 6,000 students. Suitable revisions are planned.

Other activities

1. Text materials for gifted students are badly needed. Some material of this kind is included as optional sections in the textbooks mentioned above, and other material will be published separately.

2. In another direction, an experiment is now being conducted by the Minnesota National Laboratory for the Improvement of Secondary Mathematics which will test the feasibility of a correspondence course for gifted students. This, if successful will be one way to provide for gifted students enrolled in schools too small to offer special sections or courses.

3. The sample textbooks mentioned above for grades 9 through 12 were written explicitly for college-capable students. Another project is devoted to the construction of an improved curriculum for less able students. As a first step it will test the hypothesis that such students can learn the kind of mathematics contained in the SMSG 9th and 10th grade texts, provided that the material is presented in a less formal fashion and with more concrete illustrations, and provided that the students are allowed to proceed at their own pace. Appropriate revisions for some of these texts were made in the summer of 1960 in order to carry out this test. A similar test is being conducted with the SMSG texts for grades 7 and 8. The Minnesota Laboratory is actually engaged in this experimentation.

4. Many students develop in school a negative attitude toward mathematics and hence are lost to science and technology. The SMSG sample textbooks are now being studied by a group including both mathematicians and social scientists, to see how they affect attitudes toward mathematics.

General policy on classroom use of SMSG texts

The following comments may be of some help to school authorities thinking of adopting SMSG texts.

It should be kept in mind that most secondary school teachers, through no fault of their own, were not provided in their preservice training with the mathematics which the use of these texts requires. Consequently, most teachers will need some help in the form of additional training in mathematics when teaching these texts for the first time. However, experience in the SMSG experimental centers suggests that an in-service training program, taught by a subject matter specialist, either before or during the first year's use of the texts, will be quite satisfactory in answer to this problem.

Evidence from the SMSG 7th and 8th grade centers indicates that when a teacher gives an SMSG course for the second time the need for in-service assistance and for extra time in preparation is drastically reduced, and in many cases disappears entirely.

An important source of the needed additional mathematical training is the program of summer and in-service institutes sponsored by the National Science Foundation. A number of the NSF summer institutes in 1960–61 will concentrate heavily on SMSG courses, and many others will undoubtedly use SMSG materials as supplements to their regular courses.

However, SMSG cannot itself provide any direct assistance to school systems wishing to use SMSG texts. It is a basic American principle that education is locally controlled and is independent of the federal government. SMSG, which receives all its financial support from the federal government through the National Science Foundation, wishes to do nothing which might be interpreted as an attempt to influence this local control of education. SMSG will confine its activities to the preparation and testing of improved texts. The decision to adopt these texts, and the implementation of the decision, is entirely up to local school systems.

At intervals SMSG publishes a newsletter. The function of the newsletter is to announce new projects, procedures, and policy statements, as well as the description, availability, and prices of publications. If you would like to receive issues of this newsletter, please request on a post card that your name be added to the mailing list. Address:

School Mathematics Study Group
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New Haven, Connecticut

Have you read?

ROYCE, JOSEPH R. "The Search for Meaning," *American Scientist*, December 1959, pp. 515–535.

Now and then one comes across an article completely out of the field of mathematics which still has many implications for mathematics teaching. This article is one of those. All of you have given much thought to meaning, what it is, how it is attained, and what ingredients constitute total meaning. The author gives an excellent exposition on the subject.

Man has four paths to knowledge, each containing four steps—his process, his approach to reality, his consequent understanding of reality, and the nature of ultimate reality which he does not reach. One path is through thinking, rationalism, and logic-illogic; this is scientific. Another path is through feeling, intuition, and insight-noninsight. A third is through sensing, empiricism and perception-misperception, and the fourth is through believing, authoritarianism, and ideology-delusion.

Regardless of the path, man does not cross the barrier to ultimate reality. Man therefore cannot have absolute truth—only relative truth. This fact is distressing to man; he does not want freedom of choice, and he does not want the responsibilities of decision. Meaning thus affects personality and values. As men, each of us has a value hierarchy, and this causes us to structure our value universe. Read this article twice, then try to structure its thesis in the symbolic language of mathematics. It made me stop, think and wonder.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

YASHIDA, AKIRA. "Fermat and the Theory of Numbers," *Math Student*, January 1960, p. 8.

Here is a student's article about a very famous problem. But not only that, this article gives some insight into the number system and some of the problems which were faced as our system was developing. It gives one a feeling for the complexities in the study of our number system; it shows how modern machines have helped us extend what we know about the system; and, of course, it ends with the challenge to prove that $x^n+y^n=z^n$ is impossible for x, y, and z integers $\neq 0$ and greater than 2. Students may want to work for the 100,000 mark award offered by Paul Wolfskehl, a professor of mathematics in Germany in 1908.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Edited by Howard Eves, University of Maine, Orono, Maine

Abū al-Wafā' on the solar altitude

by Nadi Nadir, American University of Beirut, Beirut, Lebanese Republic

Introduction

This paper deals with a fundamental problem of ancient astronomy, that of determining the time of day by observing the solar altitude. It is based on a short treatise by Abū al-Wafā' al-Būzjānī [1]* (fl. 970 A.D.), who is best known as having been one of the discoverers of the sine theorem in spherical trigonometry. In his introduction Abū al-Wafā' states that there has been talk to the effect that a certain rule stemming from Ḥabash al-Ḥāsib (fl. 850) is only approximate and not susceptible of proof. Abū al-Wafā' then proceeds to give three proofs of the rule. What is of more interest is that the same rule is given in a work [2] of the Hindu mathematician Brahmagupta (fl. 650). We are thus presented, not only with an incident in the emergence of trigonometry from spherical astronomy, but also with a specific example of how Hindu science penetrated that of the Islamic world.

The Arabic text of the treatise was published as the fifth in a volume of such studies, called *Rasā'ilul-Mutafarriqa f'il-hai'at*, by the Osmania Oriental Publications Bureau, Hyderabad-Deccan, India, in 1948. The work of Abū al-Wafā' is called *Fī al-burhān 'alā al-dā'ir min al-falak*.

* Numbers in brackets refer to references at the end of the article.

Notation and Definitions

Invariably in Hindu trigonometry, and almost always among the Islamic astronomers, the trigonometric functions were defined in terms of a circle with radius other than unity. Abū al-Wafā' was one of the few who recognised the utility of the unit circle in this connection.

Nevertheless, in the sequel we will find it necessary to distinguish between the medieval and the modern trigonometric functions by writing the customary abbreviations for the former with an initial capital. Thus, for example, when R is the radius of the defining circle, we have the identical relation

$$\mathrm{Sin}_R\,\theta = R \sin \theta$$

between the medieval and modern sine functions. The subscript may be omitted when there is only one possibility for the radius of the defining circle.

The term *hypotenuse of the shadow* used in Hindu mathematical works may be defined as follows. Consider a vertical gnomon of height R casting a shadow on a horizontal plane at a time when the altitude of the sun is a. In the right triangle having as legs the gnomon and the shadow, the hypotenuse will be

$$R/\sin a = R \csc a = \mathrm{Csc}\, a.$$

This is the hypotenuse of the shadow.

The *versed sine* (abbreviated *vers*, Arabic *al-jayb al-ma'kūs*, or *al-sahm = sagitta*)

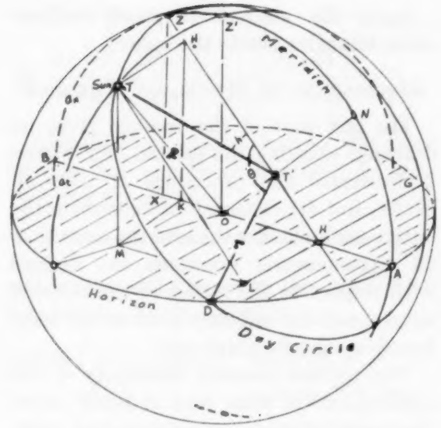


Figure 1

is defined by the identical expression

$$\text{Vers}_R\,\theta = R - \text{Cos}_R\,\theta = R(1-\cos\theta).$$

The daily path of the sun in the celestial sphere is, in general, a small circle. That part of the small circle which is above the horizon is called the *day arc* (*qaws al-nahār*). In our Figure 1 half the day arc is DTZ. Our drawing has been adapted from a figure in the text and it retains the lettering of the text, transliterated into Latin characters.

The *day sine* (*jayb al-nahār*, Sanscrit *antyā*), a concept taken over from Hindu astronomy, is the versed sine of half the day arc, $\text{Vers}_r\,\theta = HZ$, where r is the radius of the day circle. We take R as the radius of the celestial sphere.

The *arc of revolution* (*al-dā'ir min al-falak*) is the portion of the day arc from sunrise to the time in question, arc $DT=\theta-h$ in Figure 1, h being the hour angle. The arc of revolution can be interpreted as the time from sunrise to time t.

The excess of half the daylight (e, *faḍl niṣf al-nahār*) is the amount by which half the length of daylight (θ) exceeds a quarter of a day, 90° or six hours depending on the unit of time.

We use a_t to denote the solar altitude at time t, and a_n for the noon altitude. The standard symbols ϕ and δ will be employed for the terrestrial latitude and the solar declination respectively.

The Rule and First Proof

A translation of the rule as Abū al-Wafā' gives it is:

> Multiply the day sine ($\text{Vers}\,\theta$) by the sine of the altitude at the time (a_t), divide by the sine of the noon altitude (a_n), then subtract the quotient from the day sine. Find the arc versed sine of the result, and subtact from half the day arc if the observation is in the morning, but add if these measurements are taken in the afternoon. The result will be the arc of revolution ($\theta-h$).

Put into modern symbols this is

$$(1)\qquad \theta-h=\theta-\text{arc Vers}_r\cdot\left(\text{Vers}_r\,\theta-\frac{\text{Vers}_r\,\theta\cdot\text{Sin}_R\,a_t}{\text{Sin}_R\,a_n}\right),$$

in which θ, a_t, and a_n are taken as known.

With minor variants in wording the same rule indeed appears in both surviving versions of *zījes* (astronomical handbooks) by Ḥabash, in Yeni Cami (Istanbul) MS 784, 2° fol. 149r, and Berlin Arabic MS (Ahlwardt) 5750 fol. 99v.

The proof runs as follows. From T, the projection of the sun on the celestial sphere, drop the vertical line TM, and the line TL perpendicular to DG, the line of intersection of the horizon plane and the plane of the day arc. In the same manner, from Z, the sun's noon position, drop perpendiculars ZH and TL. From T draw $T\d{H}$ perpendicular to ZH and then drop $\d{H}K$ vertically from $\d{H}$.

Now, from the similar triangles TLM and ZHX,

$$TL=\frac{ZH\cdot TM}{ZX}=\frac{\text{Vers}_r\,\theta\cdot\text{Sin}_R\,a_t}{\text{Sin}_R\,a_n}.$$

Moreover,

$$\text{Vers}_r\,h=\d{H}Z=ZH-\d{H}H=\text{Vers}_r\,\theta-TL.$$

The above two expressions imply that

$$h=\text{arc Vers}_r\left(\text{Vers}_r\,\theta-\frac{\text{Vers}_r\,\theta\cdot\text{Sin}_R\,a_t}{\text{Sin}_R\,a_n}\right),$$

which leads immediately to expression (1), the rule.

We note that the procedure deals entirely with rectilinear configurations inside the sphere, in spite of the fact that the relation being investigated concerns arcs on the surface of the sphere. This technique was characteristic of Hindu spherical astronomy, as well as that of the Greeks prior to the application of Menelaos' Theorem.

The Khaṇḍakhādyaka rule says

> Multiply the *antyā* ($\text{Vers}_r\theta$) by the hypotenuse of the midday shadow and divide by the hypotenuse of the shadow at any given time; then from the *antyā* subtract the quotient obtained. The arc of the remainder taken as the versed sine represents the *asus* (a unit of time) of the incline from the noon (h).

Put into modern symbols, the rule is

$$\text{Vers}_r h = \text{Vers}_r \theta - \frac{\text{Vers}_r \theta \cdot \text{Csc}\, a_n}{\text{Csc}\, a_t},$$

which is trivially equivalent to (1).

The Second Rule and Proof

We dispense with the verbal statement of the second rule, noting only that his ignorance of negative numbers forces Abū al-Wafā' to make cases and combinations of cases for the situation when the sun is on the celestial equator, or to one or the other side of it, or when the observation is taken in the morning or the afternoon. In modern symbols the rule is

$$(2)\quad \theta - h = e + \text{arc Sin}_r \cdot \left(\frac{\text{Sin}_R\, a_t \cdot \text{Vers}_r\, \theta}{\text{Sin}_R\, a_n} - \text{Sin}_r\, e\right).$$

The proof consists of noting first that, due to the similarity of triangles ZXH and $\underset{\cdot}{H}KH$ in Figure 1,

$$H\underset{\cdot}{H} = \frac{\underset{\cdot}{H}K \cdot HZ}{ZX} = \frac{\text{Sin}_R\, a_t \cdot \text{Vers}_r\, \theta}{\text{Sin}_R\, a_n}.$$

Moreover $T'H = \text{Sin}_r e$, and $H\underset{\cdot}{H} - T'H = \underset{\cdot}{H}T' = \text{Sin}_r(\theta - h - e)$.

These facts taken in conjunction demonstrate that the right-hand side of the rule (2) can be written as

$$e + \text{arc Sin}_r\, [\text{Sin}_r\, (\theta - h - e)] = \theta - h,$$

which is the left-hand side.

Again the method of proof employs plane triangles inside the sphere.

Application of Menelaos' Theorem

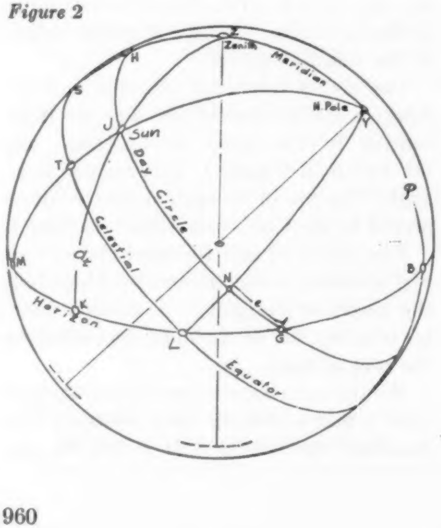
The last part of the treatise gives no drawings showing the sphere. Our Figure 2 illustrates the procedure of Abū al-Wafā', but for economy of exposition the letters are no longer those of the text. Now all trigonometric functions are defined with respect to the radius of the celestial sphere, and the ordinary symbols for these functions can be employed.

The method consists essentially of two applications of what Abū al-Wafā' called the *quadrilateral theorem* (*al-shikl al-qiṭā'*). He does not associate with it the name of Menelaos, although the connection was doubtless known to him. The surviving version of Menelaos' *Sphaerica* [3] is the Arabic recension of Abū Naṣr Manṣūr, a contemporary of Abū al-Wafā' and rival claimant to priority in the discovery of the sine theorem [4].

In Figure 2 it is required to determine arc JG as a function of a_t, the parameters ϕ and δ being assumed as known. For the spherical triangle JZY whose sides are cut externally by transversal MKB Abū al-Wafā' writes

$$\frac{\sin ZK}{\sin KJ} = \frac{\sin ZB}{\sin BY} \cdot \frac{\sin MY}{\sin MJ},$$

Figure 2



which is the same as

$$(3)\quad \frac{\sin 90^\circ}{\sin a_t}=\frac{\sin 90^\circ}{\sin \phi}\cdot\frac{\sin MY}{\sin[MY-(90^\circ-\delta)]}.$$

Abū al-Wafā' states that on this basis the arc MY can be determined. In fact, by using the identity

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,$$

which was well known to the Islamic astronomers, (3) can be written

$$(\sin MY\sin\delta-\cos MY\cos\delta)\sin\phi = \sin MY\sin a_t,$$

whence

$$(4)\quad \tan MY=\frac{\sin\phi\cos\delta}{\sin\phi\sin\delta-\sin a_t}.$$

A medieval version of the tangent function was also well known at the time the treatise was written.

The author now makes a second application of Menelaos' Theorem, taking YTS as the triangle and MLB as the transversal to write

$$\frac{\sin SL}{\sin TL}=\frac{\sin SB}{\sin YB}\cdot\frac{\sin MY}{\sin MT},$$

or

$$\frac{\sin 90^\circ}{\sin TL}=\frac{\sin(\phi+90^\circ)}{\sin\phi}\cdot\frac{\sin MY}{\sin(MY-90^\circ)}.$$

Here TL is the only unknown and can be computed, MY having been determined from (4). In degrees the arcs TL and JN are equal. Arc e can easily be computed in terms of ϕ and δ. Hence the desired arc $JG=JN+e$ can be found.

We note in conclusion that the first two proofs represent a more primitive spherical trigonometric level than the third, in the sense that the latter operates with great circles only, and entirely on the surface of the sphere. On the other hand, the third proof also is old-fashioned, to the extent that the basic configuration employed is not the spherical triangle, but the complete spherical quadrilateral, and the concept of the spherical angle does not appear at all. The sine theorem is an example of spherical trigonometry in the proper sense, but it cannot easily be applied to this particular problem.

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2. *The Khaṇḍakhādyaka* (transl. by Prabodh Chandra Sengupta), iii, 15, Calcutta, 1934, p. 75.
3. M. Krause, "Die Sphärik von Menelaos aus Alexandrien in der Verbesserung von Abū Naṣr Manṣūr b. 'Alī b. 'Irāq," *Abhandlungen d. Ges. d. Wiss. zu Göttingen, Philol.-hist.* KL., 3. Folge, No. 17, Berlin, 1936.
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The second half of the century in the history of mathematics

by Cecil B. Read, University of Wichita, Wichita, Kansas

Students and teachers are often interested in dates which might be commemorated in mathematical history. This may involve a one-hundredth anniversary, a one-hundred-and-fiftieth anniversary, or the date may be of interest purely from the point of view of how long some concept has been in existence. Histories of mathematics tend to be written in a topical order, although there are often chronological tables available. The compilation of any list of important dates in the history of

mathematics (or, for that matter, for any other topic) is largely subjective. Just what should be included is indeed debatable.

There have been several articles which give happenings during various years, but, in many cases, these are restricted to the birth and death dates of famous mathematicians. From one point of view the date of birth or date of death is of considerably less importance than the date on which the individual's principal contribution appeared. Some twenty-five years ago a sequence of articles appeared which listed as centennial years in the history of mathematics the dates from 1925 through 1936.* These articles are quite complete but only cover the years as centennial years. Moreover, all of the dates are in the first half of the century.

The accompanying list of chronological dates is divided by decades within the century. Deliberately, dates of birth and death of famous mathematicians have been excluded. Quite possibly no other author would have included or excluded the same items. Nevertheless, the suggestions may prove of value to the mathematics club wishing to commemorate a certain date, the teacher wishing an unusual or relatively unknown bit of information, or the student who wishes to pursue some topic in more detail.

The Fifties

1750 Euler uses S to denote the semiperimeter of a triangle.

1654 First known slide rule in which the slide moves between parts of a rigid frame.

1854 George Boole publishes *The Laws of Thought.*

1655 First printed use of ∞ for infinity (Wallis in *Arithmetica Infinitorum*).

1755 Euler introduces Σ to indicate summation.

1656 J. Wallis *discusses* fractional and negative exponents (without actually using them).

1856 Steiner studies the purely geometric theory of cubic surfaces.

1557 Recorde publishes *Whetstone of Witte,* first English treatise on algebra; first use of = sign.

1657 First printed work on probability, by Huygens.

1658 John Newton publishes the most complete work on trigonometry up to this time.

1758 Montucla publishes his two-volume *Histoire des Mathématiques.*

1659 First printed use of ÷ to indicate division.

1659 Johann Hudde first lets a letter stand for negative as well as positive values.

1759 Kastner first defines the trigonometric functions expressly as pure numbers.

The Sixties

860 First table of tangents and cotangents constructed by Aḥmed ibn 'Abdallâh.

1761 Lambert rigorously proves that π is irrational (published 1768).

1662 John Graunt publishes the first mortality table.

1864 Peaucellier proposes problem of generating a straight line other than by tracing a line (as by use of a ruler). His solution published 1873.

1665 Pascal publishes his *Arithmetical Triangle.*

1765 Euler proved that the orthocenter, circumcenter, and centroid of a triangle are collinear on a line now called the *Euler line* of the triangle.

1865 London Mathematical Society organized.

1866 Hamilton publishes *Elements of Quaternions.*

1667 James Gregory first uses the terms "convergent" and "divergent" series.

1569 First use of term "radius" by Ramus.

1669 Isaac Barrow (at age 39) resigns his

* Eells, W. C., "19— as a centennial year in the history of mathematics," *American Mathematical Monthly,* 32: 258–259; 33: 274–276; 34: 141–142; 35: 437; 36: 99–100; 37: 150–151; 38: 100–101; 39: 298–299; 40: 359–360; 41: 260–261; 42: 171–173; 43: 234–235.

professorship of mathematics at Cambridge to his pupil Newton.

1669 Newton gives a series for arc sin x.

The Seventies

1671 Newton writes his treatise on *Method of Fluxions.* (Not published until 1736.)

1572 Publication of an algebra by Bombelli (first use of the word "minus" in its modern sense).

1872 Klein announces his *Erlanger Programm* for the codification of geometries.

1872 La Société Mathématique de France organized.

1873 (June 5) First printed use of the term "radian."

1674 Jonas Moore first uses the symbol *cos* for cosine.

1874 Weierstrass gives an illustration of a continuous function which has no derivative.

1874 William Shanks computes π to 707 places (later discovered to be in error).

1175 Gerard of Cremona translates the Almagest.

876 Earliest undoubted occurrence of a zero (Gwalior, India).

976 Appearance of oldest definitely dated European manuscript known to contain Arabic numerals.

1676 (June 13) Newton first uses fractional and negative exponents.

1676 (June 13 and October 24) Newton generalizes the binomial theorem for negative and fractional exponents.

1477 The theory of probability is mentioned in connection with dice throwing (Benvenuto d'Imola).

1777 Euler uses i for $\sqrt{-1}$.

1478 The earliest printed arithmetic is published in Treviso (Italy).

1878 *American Journal of Mathematics* is established under the editorship of J. J. Sylvester.

1579 Publication of Vieta's *Canon Mathematicus Seu ad Triangula Cum Appendicibus.*

The Eighties

1582 Reformation of the Julian calendar under Pope Gregory XIII.

1882 Lindemann proves that π is transcendental.

1583 Fincke first uses the name "secant."

1883 Edinburgh Mathematical Society founded.

1684 Leibniz publishes his first paper on the differential calculus.

1884 Circolo Matematico di Palermo organized.

1485 First printed work on trigonometry (possibly published prior to 1485).

1585 Simon Stevin's *De Thiende,* presenting the advantages of decimal fractions.

1687 Printing of Newton's *Principia.*

1887 First exhibit of a machine performing multiplications directly by the multiplication table, rather than by repeated addition.

1888 American Mathematical Society organized (original name is New York Mathematical Society).

1489 Widmann's Arithmetic—the earliest printed book using + and −.

1789 French Académie des Sciences appoints a committee to work out a new system of measures, from whose work the metric system developed.

The Nineties

1890 Deutsche Mathematiker-Vereinigung is organized.

1890 Peano gives an illustration of a "space-filling curve."

1592 Burgi uses a comma to represent the decimal point (he also uses a period).

1692 Leibniz introduces the word "coordinatae."

1792 (September 22) Start of the new era under the French Revolutionary calendar [the Gregorian calendar was reëstablished on January 1, 1806].

1593 Vieta expresses $2/\pi$ as an infinite product.

1693 Theory of determinants begins with work of Leibniz.
1494 First evidence of permutations in print (Pacioli's *Suma*).
1494 Publication of Pacioli's work on arithmetic.
1694 First use of the word "function" (although not in the modern sense) by Leibniz and Jakob Bernoulli.
1694 Jacques Bernoulli first mentions the lemniscate.
1794 French *Journal de l'École Polytéchnique*, the first journal devoted chiefly to advanced mathematics, is launched.
1794 Vega publishes his "Thesaurus," a ten-figure table of logarithms.
1595 First appearance of the word "trigonometry" as the title of a book on the subject.
1895 Klein publishes clear, simple, and conclusive proofs of the impossibility of trisecting any angle and of duplicating a cube.
1696 First use of the term "integral calculus."
1797 Lagrange publishes his work on the calculus.
1797 Caspar Wessell presents the paper giving the earliest printed graphic representation of imaginary numbers (published 1799).
1897 First "International Mathematical Congress" at Zurich, Switzerland.
1898 *Encycklopadie der Mathematischen Wissenschaften* begins publication.

Letter to the editor

Dear Editor:

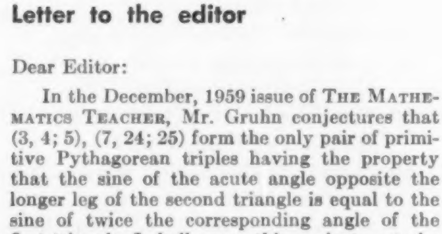
In the December, 1959 issue of THE MATHEMATICS TEACHER, Mr. Gruhn conjectures that (3, 4; 5), (7, 24; 25) form the only pair of primitive Pythagorean triples having the property that the sine of the acute angle opposite the longer leg of the second triangle is equal to the sine of twice the corresponding angle of the first triangle. I shall prove this conjecture to be incorrect.

First, however, since $\sin 2\theta = \sin (180° - 2\theta) = \sin 2(90° - \theta)$, we may state the problem more generally in the following theorem, which also solves an auxiliary problem.

Theorem:

(i) There are an infinite number of pairs of primitive Pythagorean triples $(x_1, y_1; z_1)$, $(x_2, y_2; z_2)$ having the property that the sine of the acute angle opposite the *longer* leg of the second triangle is equal to the sine of twice either of the acute angles of the first triangle.

(ii) There are an infinite number of pairs of primitive Pythagorean triples $(x_1, y_1; z_1)$, $(x_2, y_2; z_2)$ having the property that the sine of the acute angle opposite the *shorter* leg of the second triangle is equal to the sine of twice either of the acute angles of the first triangle.



In either part, if x_2 is respectively the longer and shorter leg, all such triples are of the form

$$x_1 = 2ab, \quad y_1 = a^2 - b^2, \quad z_1 = a^2 + b^2$$
$$x_2 = 4ab(a^2 - b^2), \quad y_2 = 4a^2b^2 - (a^2 - b^2)^2,$$
$$z_2 = (a^2 + b^2)^2$$

where a and b are relatively prime positive integers, one of which is even and the other odd. In part (i), when x_2 is the longer leg,

$$(\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a > b > (\sqrt{2} - 1)a.$$

In part (ii), when x_2 is the shorter leg,

$$(\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a < b < a.$$

[Roughly, $.7a > b > .4a$ and $.7a < b < a$.]

Proof:

Then,

$$\sin 2\theta = \sin 2(90° - \theta) = \sin \phi$$

$$\frac{2x_1y_1}{z_1^2} = \frac{x_2}{z_2}$$

or

$$\frac{2x_1y_1}{x_1^2 + y_1^2} = \frac{x_2}{z_2}.$$

(*Continued on page 472*)

• NEW IDEAS FOR THE CLASSROOM

Edited by Donovan A. Johnson, University of Minnesota High School, Minneapolis, Minnesota

New uses for the overhead projector

by Viggo P. Hansen, University of Minnesota High School, Minneapolis, Minnesota

One of the most effective visual aids for mathematics instruction is the overhead projector. It has great potential for adding meaning to the abstract symbols and processes of mathematics. It is a convenient, simple, and at the same time dramatic, device for illustrating and applying mathematical principles and operations.

The overhead projector will project on the screen the writing or drawing of the instructor as he is facing the class and teaching a mathematical concept. It will also project drawings or copies of any printed material such as tables, forms, graphs, or pictures. A major advantage of this device is that the instructor can place one drawing upon another with so-called overlays. This makes it possible to build an illustration as it is discussed with the class. Because the overlay drawings may be prepared in advance, correct and attractive projections can be made.

The overhead projector depends upon light passing through a transparent medium, called a transparency, interrupted only by the markings that can be made on this sheet. Through a system of lenses and mirrors, the image from this transparent sheet is greatly magnified and then projected on a screen or other light-colored surface, such as a wall. Until quite recently it was difficult to produce transparencies quickly and to add color to them. However, with the new printers now available, a copy can be reproduced on a transparency in a few minutes from almost any printed material such as textbooks or magazines. Only semidarkness is necessary, and hence the class can readily take notes while the projector is being used. Turning off the lights and projecting the lesson on the screen helps focus the students' attention on the topic being discussed.

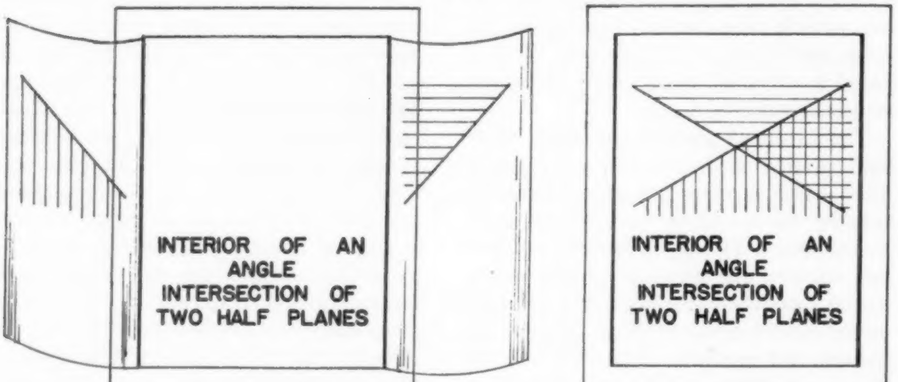
One of the most useful aspects of the overhead projector is the projection of writing or sketching by the instructor. This writing is enlarged as it is projected on the screen, so that every student can read the material. The cumbersome chalkboard instruments that are at best awkward to use can now be replaced by the conventional drawing instruments. By using clear plastic protractors and drawing instruments, the drawings can be more accurate and more easily read than board work. For example, using a transparency with grid lines permits accurate drawings of graphs. Using a combination of transparencies prepared for the lesson with comments added during the discussion gives the lesson a professional touch that is hard to attain at the chalkboard alone. Colored pencils for writing on transparen-

cies are as readily used as colored chalk. Writing on transparencies can also be erased so that a prepared sketch or graph can be used many times.

A primary advantage of the overhead projector is the use of prepared transparencies. It is not necessary to waste time during the class period to make accurate drawings. The desired transparency can be made before the class meets. Transparencies can be made by writing directly upon them or by copying some printed material. Printed material such as charts, forms, pictures, or graphs can be printed without damage to the material being copied. The process of copying is inexpensive and takes only a few minutes. By the use of foils, this copy process can be done in color, thus each overlay may be done in a separate color. An overlay consists of one transparency placed upon another transparency to add information. For example, making a series of overlays in color can be an effective way to show the intersection set of two loci or graphs. The first transparency presents the grid, a second transparency in red is used as an overlay to give one loci, and the third overlay in green contains a second loci. Placing one upon the other in the projector accurately illustrates the intersection of the two loci. This is how overlays add information as a topic is developed.

When considering the selection of topics for transparencies, capitalize on multisensory learning and work toward saving valuable time during the classroom hour. For example, introductory material on sets and Venn diagrams of the basic operations, union and intersection, lend themselves nicely to overlay transparency treatment. This same technique can be used in discussing many of the basic ideas of geometry, such as showing how the interior of an angle may be defined as the intersection of two half planes (Fig. 1). Graphing of solution sets for sentences of one and two variables can be shown by having prepared grids upon which to work (Fig. 2). The notion of a one-to-one correspondence between the real numbers and points on a line may be illustrated. The graphing of relations and functions can be compared and demonstrated side by side on the same transparency. Here there is the possibility of a great amount of time saved since these graphs can be neatly and accurately prepared before class by printing transparencies from new curricula materials. The Cartesian and polar grids can be copied from other sources and made available in the class. They are likely to be superior in accuracy to most classroom drawings made on the chalkboard. Some examples of series and the concept of limit can be reproduced and projected. An ex-

Figure 1



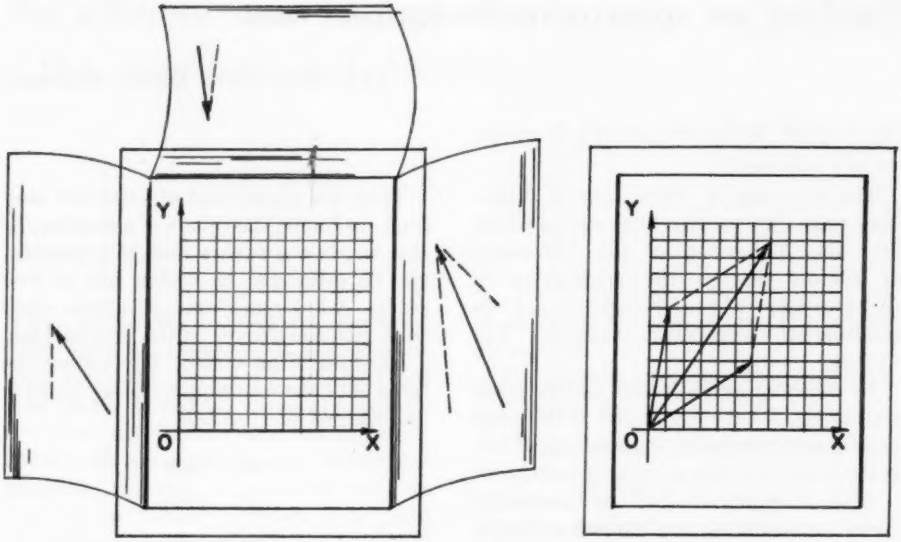


Figure 2

ample here is a set of overlays showing how the area of a regular polygon inscribed in a circle increases as the number of sides increases. The topic of vectors can be handled through a series of transparencies, as can some of the phases of probability and statistics.

In geometry and trigonometry there is literally no end to the number of transparencies that can reasonably be used. Demonstrating two-column proofs may be accomplished on prepared transparencies where the statements and reasons have been omitted and are added in the classroom as the students supply the necessary information. Even in arithmetic transparencies showing how people in the past have symbolized numbers, using a variety of colors for added effect, is a quicker and more comprehensive coverage of the topic than a few selected symbols hurriedly written on the chalkboard during the lecture. Charts comparing various number bases, and how computation is carried out in each, may add appreciably to the teacher's presentation. A study of the natural number system and its concommitant properties can be made more exciting and meaningful, and at the same time covered more rapidly by having these visual aids available to assist the regular discussion. These are but a few examples of the possibilities for this device.

The cost of repeatedly changing texts is prohibitive, but through the use of this projector the interested mathematics teacher may continue to bring to his students the latest materials. Use it to lend variety, interest, and clarity in your school. The limit to which this device can be employed in mathematics teaching is solely determined by the individual teacher's time and initiative. If your school has an overhead projector, try it. If your school does not have one, the National Defense Education Act appropriated federal funds for the purchase of materials such as this.

Finding an approximate square root

by H. Grace Baird, Racine, Wisconsin

The February 8, 1960, issue of *Newsweek* magazine described the way in which Dr. Max Beberman of the University of Illinois teaches the application of $(n+\frac{1}{2})(n+\frac{1}{2})=n^2+n+\frac{1}{4}=n(n+1)+\frac{1}{4}$ in multiplying arithmetic numbers like $9\frac{1}{2}\times 9\frac{1}{2}$.

In teaching seventh and eighth grade arithmetic, I have found one of the most helpful and interesting applications of this is the finding of approximate square roots.

After an explanation of the squaring of $(n+\frac{1}{2})$, extend it to any number ending in 5. The square will end in 25 and decimals may be placed as needed.

$$\begin{aligned} 5^2 &= 25\\ 15^2 &= 225\\ 25^2 &= 625\\ &\vdots\\ 75^2 &= 5625\\ &\text{etc.} \end{aligned}$$

It is then shown by a few examples that any number ending in 0 will have the square ending in 00. The first numerals of the square will be the square of the first numeral of the number.

Putting these together, a student very quickly has a table *in his head* or can jot down—

$$\begin{aligned} 5^2 &= 25\\ 10^2 &= 100\\ 15^2 &= 225\\ 20^2 &= 400\\ &\vdots\\ 90^2 &= 8100\\ 95^2 &= 9025\\ 100^2 &= 10{,}000 \end{aligned}$$

Thus, the square root of a number (any time one knows the square of a number, he also knows the square root of a number) can be estimated to within one or two points. A few quick multiplications could even give the answer to the nearest 10th.

Example: Find $\sqrt{4692}$. Think along the table until the following numbers near the one wanted are reached:

$$\begin{aligned} 60^2 &= 3600\\ 65^2 &= 4225\\ 70^2 &= 4900 \end{aligned}$$

Thus $\sqrt{4692}$ is between 65 and 70—a little nearer 70—try 68,

$$\begin{aligned} \text{Multiply } 68\times 68 &= 4624\text{—too small}\\ 69\times 69 &= 4761\text{—too large} \end{aligned}$$

The answer is between 68 and 69. To get the nearest tenth two or three more multiplications should give the answer. Interpolating above, try 68.5.

$$68.5\times 68.5=4692.25$$

This is so very close that 68.5 could be accepted as the square root to the nearest tenth. However, a student must learn to get one number too small and one too large and take the closest before a final decision is made. Here, then, he would be required to multiply $68.4\times 68.4=4678.46$ before the decision that 68.5 is closer is made.

Conclusion: $\sqrt{4692}=68.5$ to the nearest tenth.

I found this method much more meaningful and interesting, as well as easier, than most "division" methods taught in textbooks.

An efficient way of factoring the quadratic polynomial

Arthur E. Tenney, New Trier High School, Winnetka, Illinois

One of the big bugaboos I remember from my own high school mathematics class and from my recent teaching has been the factoring of the general quadratic by trial and error. The ninth grade SMSG text has shown how to remedy this situation. However, I would like to present a more formal approach that made my students Jack (or Jill) the quadratic killers.

First, the student proves the following two theorems:

1. If a is a factor of b and a is a factor of $(b+c)$, then a is a factor of c. (Proof: If $am=b$ and $an=b+c$ then $am+c=an$ and $c=a(n-m)$, where $n-m$ is another integer. $\therefore$ a is a factor of c).
2. If a is a factor of b and is not a factor of $(b+c)$, then a is not a factor of c. (Indirect proof).

The student is then asked to change the following three products of binomials to sums where a and b are natural numbers.

1. $(x+a)(x+b)=x^2+(a+b)x+ab$
2. $(x+a)(x-b)=x^2+(a-b)x-ab$
3. $(x-a)(x-b)=x^2-(a+b)x+ab$

It can readily be seen that the factorable quadratic, x^2+px+q with p and q as integers, can be factored only into the product of two sums, the product of a sum and a difference, or the product of two differences. (Note: The reason for making three different situations instead of one is that the sum or difference of two naturals is easier to visualize than the sum of two integers which can turn into a difference: $a+b=|a|-|b|$ when $|a|>|b|$ and $a>o$, $b<o$).

The student then does his first factorings as follows:

A. $x^2+20x+36=(x+a)(x+b)$

$ab=36=3^2\cdot 2^2$

$a+b=20$

If 3 is a factor of a and is not a factor of $a+b$, then 3 is not a factor of b. The 3^2 belongs to either the a or the b. If 2 is a factor of a and 2 is a factor of $(a+b)$, then 2 is a factor of b. The 2's must be broken up between a and b.

$a=3^2\cdot 2$; $b=2$

$\therefore x^2+20x+36=(x+18)(x+2)$

B. $x^2-16x-36=(x+a)(x-b)$

$ab=36$

$a-b=-16$

The 3's are in one factor; the 2's are broken up.

$a=2$; $b=3^2\cdot 2$

$\therefore x^2-16x-36=(x+2)(x-18)$

In easy factorings like the above, most students can do, and are urged to do, the work in their heads; the others can rely on the above patterns. However, all students could profit from the above approach in factoring $x^2+95x-5400$.

This approach doesn't really bear fruit until the general quadratic px^2+qx+r appears where p, q, r are integers and $p>o$. The three possible combinations of bi-

nomial factors where a, b, c, d are non-negative integers and $ac > o$ are:

1. $(ax+b)(cx+d) = acx^2 + (bc+ad)x + bd$
2. $(ax+b)(cx-d) = acx^2 + (bc-ad)x - bd$
3. $(ax-b)(cx-d) = acx^2 - (bc+ad)x + bd$

The coefficient of the x^2 term of any factorable quadratic is ac; the constant is bd. Their product is $acbd$ or $abcd$.

The factoring follows the pattern of the simpler quadratics.

A. $6x^2+7x-24 = (ax+b)(cx-d)$

$abcd = 6 \cdot 24 = 3^2 \cdot 2^4$

$bc - ad = 7$

The 3^2 and 2^4 must belong to either bc or ad. $dc = 2^4$, $ad = 3^2$. Since

$ac = 3 \cdot 2$, $a = 3$, $c = 2$, $d = 3$, $b = 2^3$.

$\therefore 6x^2+7x-24 = (3x+8)(2x-3)$

B. $6x^2-61x+150 = (ax-b)(cx-d)$

$abcd = 6 \cdot 150 = 2^2 \cdot 3^2 \cdot 5^2$

$bc + ad = 61$

The 2^2, 3^2, 5^2 belong to either bc or ad

$bc = 5^2$, $ad = 2^2 \cdot 3^2$.

Since $ad = 2^2 \cdot 3^2$ and $ac = 2 \cdot 3$, $a = 2 \cdot 3$, $c = 1$,

$d = 2 \cdot 3$, $b = 5^2$

$\therefore 6x^2 - 61x + 150 = (6x-25)(x-6)$

The trinomial square and the difference of two squares can be mixed in and can be factored as above. Then at an appropriate time they can be generalized.

The main support for this approach is that the students enjoyed it. By using the two theorems, much of the trial and error was taken out, but not enough to keep it from being a game. After "a" was found in the general trinomial, they found "b", "c", "d" falling like dominoes. By this time my students had enough background to multiply two general binomials. In doing so, they saw one more instance that generalized problems could sometimes lead to patterns otherwise disguised.

Letter to the editor

(Continued from page 466)

Since $(x_2, y_2; z_2)$ is a primitive Pythagorean triple, we know from elementary number theory that all values of x_2, y_2, and z_2 are given by $x_2 = 2mn$, $y_2 = m^2 - n^2$, $z_2 = m^2 + n^2 (m > n)$, where m and n are relatively prime positive integers, one of which is even and the other odd.

Therefore,

$$\frac{2x_1y_1}{x_1^2+y_1^2} = \frac{2mn}{m^2+n^2}.$$

Since both fractions are clearly irreducible, it follows that

$$m = x_1, \quad n = y_1.$$

Furthermore, since $x_1^2 + y_1^2 = z_1^2$, $m^2 + n^2 = p^2$ where $p = z_1$. Again, all values of m, n, and p are given by $m = 2ab$, $n = a^2 - b^2$, $p = a^2 + b^2 (a > b)$ where a and b are relatively prime positive integers, one of which is even and the other odd and $a > b$.

Therefore, all values of x_1, y_1, z_1, x_2, y_2, and z_2 are given by

$$x_1 = m = 2ab, \quad y_1 = n = a^2 - b^2, \quad z_1 = p = a^2 + b^2$$
$$x_2 = 2mn = 4ab(a^2 - b^2),$$
$$y_2 = m^2 - n^2 = 4a^2b^2 - (a^2 - b^2)^2$$
$$z_2 = m^2 + n^2 = (a^2 + b^2)^2$$

Since y_2 is positive, $b > (\sqrt{2} - 1)a$. In part (i) of the theorem, x_2 is the longer leg. Therefore $(\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a > b$. In part (ii), x_2 is the shorter leg, therefore

$$(\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a < b < a.$$

Hence, in part (i)

$$(\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a > b > (\sqrt{2} - 1)a,$$

and in part (ii) $b > (\sqrt{4 - 2\sqrt{2}} - \sqrt{2} + 1)a$.

In part (i) of the theorem, for $a = 2$, $b = 1$, we obtain Mr. Gruhn's pair (4, 3; 5), (24, 7; 25). For $a = 3$, $b = 2$, we obtain the new pair (12, 5; 13), (120, 119; 169).

In part (ii) of the theorem, for $a = 4$, $b = 3$, we obtain the pair (24, 7; 25), (336, 527; 625). Clearly, there are an infinite number of primitive Pythagorean triples satisfying either part (i) or part (ii).

Yours, truly
R. Beran
Toronto, Ontario

• NEWS NOTES

Continental Classroom, the network TV program for college credit, continues to focus on improving science instruction during the academic year 1960–61. The 1960–61 course, televised over NBC-TV, is Contemporary Mathematics. The first semester is devoted to Modern Algebra; the second semester, will be devoted to Probability and Statistics.

Dr. John E. Ivey, Jr., president of the Learning Resources Institute, stated the need for the program:

"A critical demand for adequately prepared mathematics teachers currently exists. Because of the shortage of teachers in this field, the mathematics curriculum in thousands of high schools and hundreds of colleges is severely restricted. Unless this shortage is alleviated, the nation's future scientific and technological progress may not keep pace with civilian and military needs.

"Contemporary Mathematics, a course designed to give new insight into the subject, can make a significant contribution by helping to alleviate the teacher shortage. In addition, it can assist in encouraging capable young people to consider mathematics as a career."

To encourage credit enrollment, a new format for Continental Classroom has been created. Modern Algebra will be divided into two sections. College and university students seeking undergraduate credit will be required to view the lessons telecast on Monday, Wednesday and Friday from 6:30 to 7 A.M. Teachers and others enrolled for graduate credit in the School of Education will be required to view the telecasts five days a week. The additional TV sessions on Tuesday and Thursday will be devoted to the teaching of modern algebra in secondary schools. The same pattern will be followed during the second semester when Probability and Statistics is offered.

"Continental Classroom hopes to stimulate interest in mathematics *now*," Dr. Ivey explains. "By so doing, it can help meet future needs for professional manpower."

Cosponsors of Contemporary Mathematics are the Conference Board of the Mathematical Sciences, the Learning Resources Institute and the National Broadcasting Company. An eight-man advisory committee, headed by Dr. E. G. Begle of Yale University, has been appointed by the Conference Board of the Mathematical Sciences.

National teacher of Modern Algebra, selected by the advisory committee will be Dr. John L. Kelley, professor of mathematics and head of the mathematics department at the University of California, Berkeley. Present plans indicate that Dr. Frederick Mosteller of Harvard University will teach Probability and Statistics. For the teacher-education programs of each course, an outstanding high school instructor will assist the national teacher.

As outlined by Dr. Kelley, Modern Algebra will include the fundamental concepts underlying the recent changes and developments in both the approach to and teaching of this subject. It will supply all prerequisites for the second semester offering in Probability and Statistics.

"A great discovery solves a great problem but there is a grain of discovery in the solution of any problem."—*G. Pólya in the Preface to* "How to Solve It."

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Basic Geometry, George Birkoff and Ralph Beatley (New York: Chelsea Publishing Company, 1959). Cloth, 294 pp., $3.95.

There have been many articles written recently pertaining to plane geometry texts, particularly about the omission of important concepts such as betweenness, co-ordinate geometry and solid geometry. *Basic Geometry* includes these concepts and many others that are frequently mentioned, but the manner in which the authors have presented these concepts is decidedly different from any other geometry text that I have read. The entire development of the geometry is different from the development presented in other geometry books, due to the set of original assumptions upon which the geometry is based. The authors have based the development of their geometry on properties of the real numbers. This enables them to include an original assumption pertaining to the similarity of triangles which is usually found in the latter part of most texts on plane geometry. Similarly, the notion of betweenness of points on a line follows from the properties of real numbers.

The authors do not make a distinction between a line segment and its length or between an angle and its measure. Neither do they have a set of theorems pertaining to congruences of line segments, angles, or triangles. In fact the concept of congruence is not used, since the authors have developed their geometry independently of any idea of motion. The advantage of such a development is that many proofs that are normally difficult become relatively simple, and the student is able to move rapidly to some of the more important theorems. For example, the Pythagorean Theorem is the seventh theorem proved, and thus may be used to great advantage throughout the book.

The organization of the book is also different from other geometry texts. There is no section devoted to definitions, nor is there a section at the beginning on constructions with straightedge and compasses. There is a set of undefined terms before the statement of the assumptions, but the remaining definitions are presented as they are needed. The constructions appear later, after the student has acquired sufficient facts to enable him to discover his own rules and prove them. At first glance, it seems that many of the more important theorems have been omitted, but the authors have left them as original exercises for the student to prove; thus the exercises play an important part in the development of the geometry. Exercises that are essential to the development of the geometry are starred.

The authors also make use of the exercises to introduce solid geometry. For example, after exercises on similar triangles, the student is asked to draw an equilateral triangle on a sphere and measure its angles. He is then asked to draw a larger equilateral triangle such that the triangle reaches from the pole to the equator, and then measure the angles of the triangle. From such exercises the student has an opportunity to make generalizations, and to compare and constrast plane and solid geometry.

The proofs given in the book are in a discussion form rather than in a statement-reason form. To help the student understand why certain steps are taken and why certain facts are needed, most proofs contain an analysis of the problem; therefore students should find it much easier to prove original exercises. Perhaps the material could be more easily located if the chapters were subdivided into sections, because at times it is difficult to determine when one set of exercises ends and the next section begins.

Another important difference from conventional texts is that the parallel postulate is not taken as one of the fundamental assumptions. It is proved as a theorem. The rectangular coordinate system, the concept of the slope of a straight line, and the equation of a line are introduced following the proof of "the parallel postulate," but there is little use made of these concepts. The formula for finding the distance between two points should be introduced and then these concepts should be used to prove theorems that follow in the book. For example, the above concepts could be used to prove that sides of a parallelogram are equal.

The first chapter is devoted to the nature of proof. It is a matter of speculation whether a student would really understand the nature of proof after he has read the chapter, especially the sections on the converse of a statement, errors in reasoning, and the types of proof. The contrapositive of a statement and its usefulness in proving statements is not mentioned in the book. Some treatment of symbolic logic would also help in presenting the nature of proof. A feature of the chapter that does help the student understand the nature of proof is the illustration of the types of proof.

In general, I feel that *Basic Geometry* is in accord with the present approach to plane geometry. It offers a sound mathematical development of plane geometry, and at the same time enables the student to move rapidly into the heart of geometry.—*Thomas Denmark, Florida State University, Tallahassee, Florida.*

Differential and Integral Calculus, James R. F. Kent (Boston: Houghton Mifflin Company, 1960). Cloth, xv+511 pp., \$6.75.

With the proliferation of calculus texts now available, new approaches to the subject at the elementary level are difficult to find. This text features an emphasis on explanatory material from which students should be able to achieve a better understanding of the principles of calculus than from many other texts which rely on the instructor to fill in the gaps in understanding. An illustration of the technique often employed is provided by defining in nonrigorous fashion the limit of a sequence as a unique number associated with a sequence to which, after some stage in the sequence is reached, all the terms thereafter are arbitrarily close. The author then elaborates on the meaning of this definition through examples and an ϵ, N formulation. The limit of a sequence is utilized to define the limit of a function as an alternative to the ϵ, δ definition, although this definition is included in a footnote. In fact, where gaps in logical rigor occur elsewhere the author seems to have been careful to note this. After the introductory chapter the statements of theorems and their proofs are more carefully given.

Rolle's Theorem, the Law of the Mean and its generalization, together with Taylor's formula with the remainder, are introduced prior to taking up differentials. This allows consideration of the accuracy of replacing $\Delta f(x)$ with $df(x)$ and also provides proof of the derivative tests for identifying relative extreme values of a function on an interval. Rolle's Theorem is stated in its generalized form requiring $f(a)=f(b)$ rather than $f(a)=f(b)=0$. This makes the proof of the generalization of the Law of the Mean simpler.

The text covers most of the material usually found with some topics relegated to exercises (for example, the hyperbolic functions). A chapter on solid analytic geometry precedes the section on partial differentiation and multiple integration. A synopsis of material from elementary mathematics is provided in the appendix for reference, as well as tables of values for the trigonometric and exponential functions. Answers are provided for the odd-numbered exercises with the remaining answers available to instructors upon request.

To the writer, an especially pleasing feature is the obvious care taken in construction of the exercises. Many of them are designed to reinforce and expand understanding of the theoretical principles. Where practice is needed, enough exercises are provided, but one is not surfeited with them. Many exercises are pointed toward applications from the social science and business fields, more so than in most other calculus texts with which the reviewer is familiar.

This is a text that many instructors will find to their liking. It is carefully written and, your reviewer believes, capable of being understood by students. Its stress upon the principles of calculus makes it a welcome addition to the long list of calculus texts.—*Lester R. VanDeventer, Eastern Illinois University, Charleston, Illinois.*

Elementary Analysis, A Modern Approach, H. C. Trimble and F. W. Lott, Jr. (Englewood Cliffs, N. J.: Prentice-Hall Inc., 1960.) Cloth, xii+621 pp., \$6.95.

For most teachers of freshman mathematics (algebra, trigonometry, and analytic geometry), the desire to use many of the concepts of modern mathematics as an aid to more effective presentation of their material has been thwarted by the lack of a carefully prepared textbook. Because of the time required to write their own materials for instructional purposes, many instructors rely on the textbook as a major instructional aid. Thus, to meet the need for texts which employ modern methods and utilize modern concepts, many books are in production. Many will be discarded because often the inclusion of modern concepts is done at the expense of other important concepts necessary for the study of calculus.

Elementary Analysis, A Modern Approach cannot be condemned on this score. Those believing that manipulative skills cannot be developed simultaneously with techniques of proof are invited to read Chapter II, which deals with the real numbers (see pages 56 to 62). In this chapter the student, following a discussion of the distributive principle for real numbers, is challenged to prove many of the factoring theorems. As a contrast and an illustration of the attention given to applications in this text, a section of Chapter VIII (page 387), discussing the use of logarithms in computational work, shows how proper organization of a computing form lessens the time and effort involved in a problem. Several alternate forms on the same problem let the student decide on the better form. The emphasis in *Elementary Analysis, A Modern Approach* is on doing by the student, but only after thinking has first been done. The staff of the mathematics department at Iowa State Teachers College has used this text for two years, and often the classroom technique shows in the way students are motivated to understand a difficult concept by challenging examples and illustrations preceding the formal presentation.

Elementary Analysis, A Modern Approach covers in an unusual way the usual areas of second-year algebra, college algegra, trigonometry, and analytic geometry, as well as additional material. The sequence of topics is not presented by the above classification scheme. As an example of this unusual grouping of topics, Chapter IX, Trigonometry, might be noted. This chapter includes trigonometry of the triangle, inverse functions, polar co-ordinates, DeMoivre's Theorem, identity, and conditional equations. This ordering impressed the reviewer as more conducive to learning than the placement of DeMoivre's Theorem as a topic in college algebra and polar co-ordinates as a topic

of analytic geometry—the more customary placement. Radians, degrees, revolutions, and mills are handled side by side, which should lessen the importance students place on degree measure instead of real (radian) measure.

The use of set theoretical language and concepts serves to unify the presentation in this book. One sees in studying *Elementary Analysis, A Modern Approach* how an important idea, the ordered pair concept, for example, is used in many ways in beginning mathematics. The rationals are obtained as ordered pairs of integers which were themselves obtained as ordered pairs of natural numbers. The complex numbers again utilize the ordered pair concept. Relations and the special subclass, functions, are defined by using ordered pairs; points in the plane are also identified with ordered pairs. Other set ideas are introduced in the text when they are useful to the exposition. *Parameter* is defined on page 99, using set language. This definition follows an explanation which makes it difficult to see how the idea of parameter ever caused trouble.

Logical concepts and methods of proof are stressed early in the course, and some concepts are developed which are seldom included in early mathematics work. Statements, open sentences, variables, range of the variable, the null set, properties of number systems (such as closure, associativity, etc.) all are presented in the first several chapters. In the first chapter the distinction is made between cardinal and ordinal numbers. Properties of the real numbers are given in Chapter II and theorems are proved, followed by many good exercises for students to prove. Vectors are used many times in the book, matrices are introduced, and an inversion method and applications given. The chapter on co-ordinate geometry includes locus, geometric proofs by co-ordinates, and solid analytic geometry, but more significantly discusses Euclidean geometry as characterized by rigid motions in the plane. In particular, inversion in the circle is shown to preserve magnitude of angles. This topic is normally found in college geometry at a more advanced level, though the discussion here is straightforward and comprehensible to the freshman mathematics student.

A careful reading of this book discloses few mistakes and excellent exposition. Many interesting new problems are presented (see for example 155:3, 345:15 or 575:13. The main disadvantage of the text lies in its length. In the opinion of the reviewer, it would require at least ten semester hours to cover the material. Suggestions for some possbile exclusions are made in the appendix by the authors.

This book stands out in its area and should find wide acceptance in high schools interested in advanced placement programs and in colleges as a calculus-preparatory course. If the pleasure the reviewer found in reading this textbook is shared by the student, the instructor will have found a welcome aid for his teaching. —*Archie Lytle, Central Michigan University, Mount Pleasant, Michigan.*

Logic in Elementary Mathematics. Robert M. Exner and Myron F. Rosskopf, New York, (McGraw-Hill Book Co.: 1959) cloth, vi +274 pp., $6.75.

The authors state that their purpose is threefold: to present portions of symbolic logic useful in elementary mathematics, to use logic in discussing formal aspects of mathematics, and to present the material in a manner accessible to students with elementary mathematics but not logic as a background. Chapter I is an informal introduction on the nature of mathematics and logic. Chapter II considers the statement (or sentential) calculus. Chapter III is devoted to a discussion of proofs and demonstrations with emphasis on indirect proofs, proof by elimination and by cases, and converses and inverses of theorems. Chapter IV deals with abstract mathematical systems. Special consideration is given to an axiomatic treatment of a "miniature" geometry with exactly ten points and ten lines. A purely symbolic development of this geometry is provided in an appendix.

Chapter V treats the restricted predicate calculus. A set of rules for quantifier inferences with restrictions to avoid invalid arguments is the focus of the chapter. The set of rules yields yet another system of natural deduction to be added to the many already published. (The large number of variant systems in the literature indicates that a fully satisfactory system has still to be found.) Chapter VI sets forth a number of applications of logic in mathematics centered around an explicit consideration of proofs in the elementary theory of groups and fields.

The authors are to be commended for the painstaking care with which the book has been written. I liked particularly the third chapter on proofs, the system of miniature geometry in the fourth chapter and the explicit treatment of numerous algebraic questions in the final chapter.

A few minor criticisms may be mentioned. The discussion of logical validity is sketchy and occasionally misleading in Chapter I and elsewhere. In particular, the criteria of soundness and completeness for a set of rules of inference are not adequately mentioned. The chapter on applications of logic in mathematics could well have included an intuitive sketch of the notions of computability and decision procedures without entering into technical details, for it is in this direction that many serious mathematical applications of logic have been made in recent decades. A section or chapter on the theory of definition would not have been amiss, but this opinion perhaps reflects the personal taste of the reviewer.—*Patrick Suppes, Stanford University, Stanford, California.*

Mathematics, First Course; *Mathematics*, Second Course, John A. Brown, Bona Lunn Gordey, and Dorothy Seward (Englewood Cliffs, New Jersey: Prentice-Hall, Inc., 1960). Cloth, First Course, 323 pp., $3.64; Second Course, 365 pp., $3.40.

Both of these texts provide a great advance in the teaching of mathematics in the seventh and eighth grades. Each book gives the teacher a wealth of material, which includes an appendix that offers enrichment. "Chapter Tests," "Review Tests," the "Quick Quiz," "Fun with Numbers," and "Maintain Your Skills" are section headings that give the spirit of well-organized subject matter. The pages are attractive, and where color is used, it serves a purpose.

The written form for problem solving is excellent. Even in drill exercises in computation, an estimate is required first. The neat arrangement of work with equality symbols in a vertical column and with auxiliary computation to the right of the main procedure are illustrated and can be required of the student. The answer to a problem is stated in a simple sentence. Secondary science and mathematics teachers will welcome students thus trained in writing solutions. The use of the inductive method of reasoning to develop understanding is generally employed.

Both courses start with chapters on the structure of a number system. Writing numbers to different bases is skillfully developed, reviewed, and used throughout each text.

The simple equation is introduced to the seventh-grader in teaching the three cases of percentage. On page 168 of this text, the authors state, "*p* is used for perimeter and *s* for a side." Certainly the authors should not wish to convey the idea that letters are symbols for words, and a correction will be made.

Algebra as such is introduced in the Second Course. The commutative, associative, and distributive properties of numbers are really taught and applied in this text. The number line and graphing in one and two dimensions are introduced in the first text, but since an inductive approach is not employed, the reviewer wonders if its inclusion is wise. The language of sets begins on page 28 of the Second Course and is used and reviewed at intervals throughout the text. When the later chapters on Coordinate Geometry and Topics in Algebra appear, the student is equipped to use the idea of a set in finding solution sets for equations and inequalities and in graphing. These chapters are very well done.

The work in geometry includes simple properties of the triangle and an introduction to congruent and similar triangles. Ratio as a concept is introduced in the first text, and problems involving proportion are found throughout the Second Course. The treatment of measurement of areas and volumes is satisfactory.

A chapter on probability appears at the end of the Second Course. This is excellent, but the presentation seems much too sophisticated. Having taught these ideas to high-school students, the reviewer doubts that probability should be defined before the student gains a degree of insight into the nature of the laws of chance by first performing experiments. The work on permutations and combinations is well done since the inductive approach is used. In fact, if this chapter were used only to have the student explore and to enjoy himself, the results would be excellent.

There are two ways in which the texts are inconsistent. Rules for rounding numbers on page 53 of the First Course disagree with the treatment on page 110 of the Second Course. The reviewer also suggests that the authors use the correct form for the multiplication of fractions on page 110 of the First Course. The inductive approach is excellent, but it is not good to write $\frac{1}{3}\times\frac{3}{4}=\frac{1}{4}$, showing cancellation of the threes, with the following comment: "The 3 in the numerator and the 3 in the denominator were both divided by 3." The 3's are in *a* numerator and *a* denominator, and unless the student first writes $\frac{1\times 3}{3\times 4}$ as his product, he is not apt to recognize that he is reducing *a fraction*. The second text reteaches this process in rigorous fashion.

Both texts do an excellent task in developing a vocabulary in mathematics. As a whole, language seems to be well used. However, the first text does not define *digit;* consequently, the reviewer doubts if digit, numeral, and number are really clarified for the student. On page 49 is written, "The place a digit occupies in the numeral determines its value." Does a digit have value or does it indicate value? Our ten digits are *valuable* to the extent of being priceless because each can be used to indicate so many different numbers. Throughout mathematics, digits *by their position* tell us what to do. Would it not be clearer for a seventh grader if we stated that two beads *placed* on the second string indicate the number 20? Then on page 9, "Each digit represents a certain value by virtue of its position" could be stated, "The digit 2 *placed* in the second column indicates the number 20." Since the term *positional notation* is used on page 136 of Course Two in teaching scientific notation, why not use "principle of position" to replace "principle of place value" in the first book? The second text clearly defines "place value" on page 3, and the reviewer approves of thus enlarging the concept previously taught.

Whether or not the texts use "ragged decimals" is debatable, since it is clear that they are working with exact data when they appear. On page 141 of the First Course, the text states, "Add zeros to the dividend if necessary." Zero is the Additive Identity Element, and if the authors are to make use of their idea of "place value," this seems incorrect. Is not "annex" the correct word? The reviewer wishes that computation with approximate data had been placed at intervals throughout the Second Course. So many problems of each text really give data of this nature. The idea of a "correctly recorded measurement" and its use in computation seems most difficult to teach in the high school. Is this because, in his earlier training, the student has acquired prejudices, the origin of which stems from the fact that he has never been

made aware of the difference in meaning of the questions, "Is .8 equal to .80?" and "Is .8 in. equal to .80 in.?" Zeros should never be annexed in addition or subtraction of measurements. On page 81 of the First Course is written, "The value of twelve has not been changed by *adding* two zeros." In the problem given, 12 inches and 1.75 inches are measurements of rainfall. If the authors wish to teach the subtraction of 1.75 from 12.00, the reviewer wonders if the minuend should not be given as 12.00 inches. Then, when approximate data are used, the student may be less confused when faced with, "Is 12 inches the same as 12.00 inches?" Only two pages are devoted to computation with approximate data in the chapter entitled, "Estimating and Central Tendency" in the First Course. The reviewer finds none in the Second Course. She raises the question whether this is wise. She believes that work in co-ordinate geometry should not take precedence over it. On page 112 of the Second Course, the weights of 11 football players are given. All recordings are integers or to the half pound. Yet the average is stated as being $163\frac{7}{11}$ pounds! Faced with either a football team or scales, the reviewer again wonders if this problem should be used to teach skills with exact data.

These texts make a significant contribution to junior high mathematics. The reviewer feels that a teacher should be advised to use the Second Course in accelerated classes first. The First Course should prove successful with average students. Teachers using either should know the philosophy of modern mathematics and have an excellent foundation in its approach to familiar subject matter.—*Laura M. Wagner, Senior High School, Fort Atkinson, Wisconsin.*

Mathematics in Everyday Things, William C. Vergara (New York: Harper and Brothers, 1959). Cloth, xi + 301 pp., $3.95.

The author of this book discusses a variety of mathematical ideas by means of approximately 140 questions and answers presented in deliberately random sequence. This form is like that used in his *Science in Everyday Things*. In the current book fewer questions have been asked, but each answer is more complete and satisfying to the reader since the mathematical basis has been elaborated. Because of the arrangement of questions, this is not the kind of book one is apt to enjoy reading in a continuous fashion; rather one tends to leaf through it, noting the questions, and reading those answers for which one's curiosity is piqued at the moment. References to specific topics must be made through the comprehensive index since there is no table of contents.

At least three types of questions are considered:

1. Those which deal with mathematical vocabulary; e.g., What are imaginary numbers? What is a googol? The answers to such questions are not mere definitions; often they include examples, describe applications, and illustrate possible extensions of the concepts.

2. Those which apply mathematical principles to scientific inquiry, e.g., Do any rivers flow uphill? Why does the whistle of a train seem to change pitch as it travels by us? In the answers to such questions mathematical relationships are pointed out and generalized into formulas, and a specific instance is often described.

3. Those which ask who first developed some important mathematical concept or which in other ways link the name of some personage to a concept, e.g., Who first measured the size of the earth? What is the great theorem of Fermat? The answers, while presenting some historical information, emphasize the mathematical concepts involved.

The reader who is especially interested in precision of language will find a few statements to which he may object. For example, in answering the question "What are exponents?" the following statement is made: "While we have illustrated this fact for the number 4, the same reasoning will show that any number raised to the zero power is 1: $n^0=1$." The exception, $n=0$, is not mentioned.

The random arrangement of the questions may be disconcerting to some readers. For example, the method of finding height by the law of tangents is discussed earlier than a relatively more simple method involving an isosceles right triangle.

The Doppler effect is a topic which will illustrate the scope of much of the discussion. In *Science in Everyday Things* the question of apparent change of pitch was discussed descriptively. In the current book pertinent formulas are developed, and with the help of a ratio of frequencies it is shown that a horn emitting 400 vibrations per second approaching an observer at 90 feet per second will appear to have its pitch lowered about 3 half tones as it passes. The extension of the idea of the Doppler effect in sound to spectra is presented in connection with a discussion of the composition of the sun, and here it is used to support the idea that the universe is expanding.

This book should be extremely valuable to teachers, since it gathers together for the classroom much excellent enrichment material which is normally widely scattered. Students from junior-high age upward whose interest in mathematics is just awakening will find these questions and answers fascinating, and the mathematics not too difficult to appreciate.—*Herbert F. Miller, Northern Illinois University, DeKalb, Illinois.*

• **TIPS FOR BEGINNERS**

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan, and William C. Lowry, University of Virginia, Charlottesville, Virginia

Using number theory to foster interest in general mathematics

by William M. Fitzgerald, Eastern Michigan University, Ypsilanti, Michigan

The theory of numbers offers a great deal of potential for enriching our mathematics curriculum at all levels. Some of these topics can be used to inspire the below-average students as well as the average and bright students. An interesting example is reported by Goodman,* who used children's blocks to demonstrate the basic ideas of factors and primes.

Learning theory suggests that abstractions can be formulated, especially by slower students, only through concrete and semiconcrete experiences. The purpose of this article is to describe a sequence of experiences to aid the students in developing some abstractions from number theory. This sequence was received with enthusiasm in both a general mathematics class and an eighth-grade mathematics class.

Each student must have a sheet of graph paper, a straightedge, and a pencil. Students are asked to draw a rectangle which contains exactly 36 small squares. Undoubtedly, they will produce several answers. If the sides of the rectangle are restricted to integral lengths, there are only five answers, as shown in Figure 1. A rotation of 90° gives four more rectangles, but they are not considered as different.

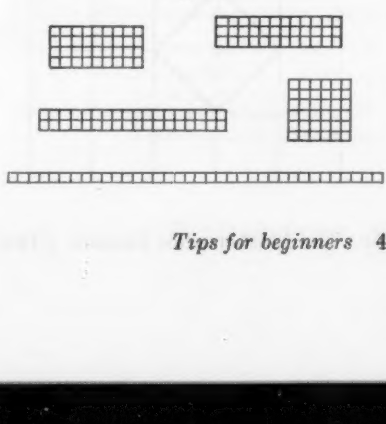
* Frederick L. Goodman, "Prime Numbers and Factoring," *The Arithmetic Teacher*, VI, (November, 1959), 274–275.

Then the students are asked to use the same procedure to find the number of rectangles with areas of 1, 2, 3, 5, 7, 10, 14, 17, 20, and 100, respectively.

These are designed to help the students discover the generalization that there are as many rectangles for any given integer as there are ways of factoring that number into whole numbers, without regard to order. The students may then be able to understand a second generalization—that positive integers can be put into three classes. The first class consists of only the number $\{1\}$. The second class consists of all numbers which can determine the area of a rectangle in only one way. These are the *prime* numbers—$\{2, 3, 5, 7, 11, 13, \ldots\}$. The third class of numbers consists of those which determine the area of more than one rectangle. These are the numbers $\{4, 6, 8, 9, 10, \ldots\}$, which are called *composite* numbers.

At this point it is possible to explore several related ideas, such as drawing circles,

Figure 1

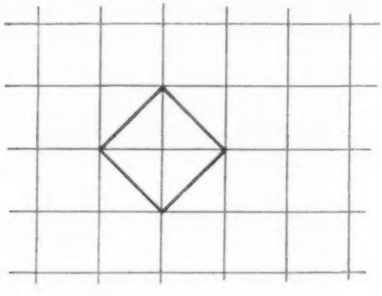


triangles, and other polygons. As an example, since the students have already drawn rectangles with areas of 36, ask them to draw a triangle with an area of 18. Or have them draw a circle in which they think they have approximately 100 small squares. These problems can add reinforcement to the skill of using area formulas.

The students may notice some special numbers in the class of composite numbers which determine rectangles with the same length and width. These numbers are called square numbers, or perfect squares. So it is very easy for students to draw squares with areas of exactly 4, 9, 16, 25, etc. Challenge them to draw a square with exactly 20 small squares in it. Some may give up. Others, being more persistent, may try a square whose side is $4\frac{1}{2}$. Upon examination, this area of this square is found to be $20\frac{1}{4}$. No matter how little they shave off or add on in this fashion, the area won't result in exactly 20 small squares. After some struggle with this problem, they begin to appreciate the nature of an irrational number such as $\sqrt{20}$. Since $\sqrt{20}$ is irrational, it cannot be expressed as the ratio of two integers. As the student continues to reconstruct the square using different lengths such as $4\frac{1}{4}$, $4\frac{3}{8}$, $4\frac{7}{16}$, $\ldots$, he can come closer and closer to a square with an area of exactly 20, but never quite reach that point.

The problem of drawing a square whose area is a nonsquare integer is solvable for many integers with our crude equipment.

Figure 2



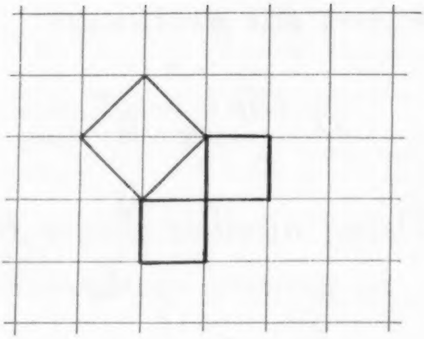


Figure 3

Let's begin again and this time try to draw a square whose area is exactly 2. The student who is pursuing the solution may discover that a square which is tipped as in Figure 2 contains exactly 4 half-squares, or 2 squares.

They may recall, with some hints from the teacher, that if they add to the picture as in Figure 3, the result will be an example of the Pythagorean theorem.

The student can now draw more squares with integral areas simply by drawing right triangles with integral sides and constructing the square on the hypotenuse. The triangle can be drawn easily by connecting any two lattice points, that is, points at the intersection of the cross-lines. The line segment can be considered to be the hypotenuse of a right triangle. The squares on the sides of the right triangle will now be integral; and the square on the hypotenuse will be the sum of the two integral squares, so it must also be integral. The general solution of this problem can be stated as follows: A square with a particular integral area may be drawn on the graph paper if the integer can be expressed as the sum of a pair from the set of squares 0, 1, 4, 9, 16, $\ldots$. We could even construct an addition table to show all the possible solutions as shown in Figure 4. All the numbers in the table represent areas of squares that can be drawn by this method.

The bright or eager student may wish to construct squares whose areas don't ap-

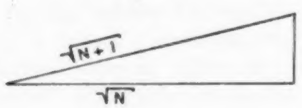
+	0	1	4	9	16	25	36	...
0	0	1	4	9	16	25	36	...
1		2	5	10	17	26	37	...
4			8	13	20	29	40	...
9				18	25	34	45	...
16					32	41	52	...

Figure 4

pear in the table, such as a square with an area of 3. With the aid of a compass, some of the distances may be transformed and perpendiculars may be erected to allow squares with any integral area to be constructed. In fact, this can be proven nicely. We know we can construct a square with an area of 1. We also know that if we have a square whose area is N and whose side is $\sqrt{N}$, we can construct a right triangle which has a hypotenuse of $\sqrt{N+1}$ so the area of its square is $N+1$. This general triangle is shown in Figure 5.

In other words, if we can find a square whose area is 1, we can find a square whose area is 2; and if we can find one whose area is 2, we can find one whose area is 3. If we can find a square whose area is N, we can find one whose area is $N+1$ for any natural number N. This is a relatively simple illustration of the principle of mathematical induction.

Figure 5



It is desirable to have a grid painted on the blackboard to illustrate these drawings. If one is not available, a piece of pegboard, some golf tees, and rubber bands would work well. The students can use pins and rubber bands on their graph paper instead of a pencil.

By allowing each student to be actively engaged in solving his own problems, the method helps provide for the wide range of ability which exists within any class. Each student will develop his own generalization based on his own experience and insights. The students have come into contact with a wealth of mathematics; and the more able ones may be encouraged to take a closer examination of these intriguing topics.

We know mathematics holds intrinsic interest for most bright students. As teachers of mathematics, we have ourselves felt the fascination of learning concepts of our number system. If the less able student can also experience fascination when working with the abstractions in the study of numbers, he may have a more favorable attitude toward all of mathematics and toward the need to develop his mathematical skills. While learning mathematics, he too should be able to say, "This is fun!"

Letter to the editor

Medford, Massachusetts

Dear Editor:

On page 119 of the February, 1960, issue of THE MATHEMATICS TEACHER is an article on the dissecting of a square.

A proof is given that purports to prove that an equilateral triangle has been formed by the given dissection of the square. My reading of this proof makes me believe it is in error.

On page 123, where quadrilateral 3 is being compared with quadrilateral 3′, it is shown that the corresponding angles are equal. However, only one pair of sides is shown to be equal and that is not a sufficient condition for proving those quadrilaterals congruent.

If this argument of mine is correct, the proof is faulty.

Respectfully,
ENOR E. LUNDIN
Master in Mathematics,
Boston English High School

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Christmas Meeting in Tempe, Arizona December 27-30, 1960

The Nineteenth Christmas Meeting of the National Council of Teachers of Mathematics will be held on December 27–30, 1960, in Tempe, Arizona, at Arizona State University.

General session speakers will include Howard F. Fehr, T. M. Stinnett, Leland Carlson, and J. D. Williams.

Mathematics lectures will be given by R. A. Dean, Burton Jones, and Raymond L. Wilder.

Elementary school sections will feature a fifth grade class demonstration by David Page, the director of the University of Illinois Arithmetic Project, and a first grade class demonstration by Patrick Suppes, Stanford University, entitled "Sets and Numbers: An approach to Arithmetic via Set Theory."

Participants from the School Mathematics Study Group elementary mathematics writing team will discuss the fourth grade text and the experimental units for grades five and six. Other talks will deal with the Madison Project, the preparation of special teachers of mathematics, and research in elementary school mathematics.

Junior high school sections will feature the SMSG seventh and eighth grade courses and the texts for the middle group in grades seven and nine. Other talks will deal with the bridge from arithmetic to algebra, problem solving, and the teaching of percentage.

Senior high school sections will review the Developmental Project in Secondary Mathematics at Southern Illinois University, the University of Illinois Project, the Ball State Geometry Program, the coordination of secondary and college mathematics, institutes for mathematics teachers, and a progress report of the SMSG for grades 9–12.

The college and teacher education sections will hear about a special program at Purdue University, an experimental seminar in science teaching at the University of Arizona, and revised college curricula. The training of teachers in statistics, geometry in the grades, demonstration computers for high schools and colleges, background mathematics for elementary teachers, and the effect of class size on college students' success will also be discussed.

Discussion groups considering the use of NDEA funds in equipping mathematics classrooms will be held at each level. Laboratory sessions at each level will provide a sharing of ideas to implement mathematics teaching.

The supervision section will discuss the services of the state supervisor of mathematics which are available to schools.

Films and filmstrips will be previewed.

Plan orientation conferences on new mathematics programs

Professor Phillip S. Jones, NCTM president, has announced a Council-sponsored plan for sending consultants into the field to help school systems strengthen their mathematics programs through the use of the improved materials of instruction now available. The project, supported by a \$48,000 grant from the National Science Foundation, is aimed at providing selected supervisors and school administrators with the information and orientation they need to provide leadership in establishing improved mathematics programs in their school systems.

With this end in view, a series of eight regional invitational conferences has been scheduled during the fall of 1960. Each conference will be under the jurisdiction of a regional director, and all conferences will be conducted by the same team of three consultants. This team consists of Dr. G. Baley Price, professor of mathematics, University of Kansas, who has an active interest in the improvement of school mathematics; Dr. Kenneth E. Brown, mathematics specialist, U. S. Office of Education, who has made a nation-wide survey of the results achieved by the use of texts recently produced through the joint efforts of mathematicians and teachers; and Dr. W. Eugene Ferguson, head of the department of mathematics, Newton High School, Newtonville, Mass., who has had experience with several of the new mathematics programs as well as with the problems of in-service training preceding such programs.

Each two-day conference has a four-part program: (1) "Progress in Mathematics and its Implication for the Secondary School," Prof. Price; (2) "The Drive to Improve School Mathematics—Comparisons and Common Elements of Special Programs," Dr. Brown; (3) "Our Experiences with the New Programs in Mathematics"—panel of teachers to be selected in each region; (4) "Implementing the New Mathematics Program in Your School," Dr. Ferguson. In addition, time will be allowed for question sessions and for the inspection of the sample texts which will be on display. A preview of this program was presented at the Council's summer meeting at Salt Lake City, August 22, 1960.

The Project is being directed by Frank B. Allen, member of the Council's Board of Directors and chairman of the mathematics department of Lyons Township High School and Junior College, La Grange, Illinois, with the advice of a steering committee of mathematicians and mathematics educators including: Dr. Max Beberman, director, University of Illinois Committee on School Mathematics, Urbana; Dr. E. G. Begle, director, School Mathematics Study Group, Yale University, New Haven, Conn.; Dr. Brown; Dr. Edwin C. Douglas, chairman of the mathematics department, Taft School, Watertown, Conn.; Dr. Jones; Dr. John R. Mayor, director of education, American Association for the Advancement of Science, Washington, D.C.; Dr. Price; and Dr. Mina Rees, dean of faculty, Hunter College of the City of New York. Ex officio members are Dr. Philip Peak, assistant dean, University of Indiana, Bloomington; and Dr. Henry Van Engen, professor of education, University of Wisconsin, Madison.

Consultants to the committee are Dr. Harold P. Fawcett, department of education, Ohio State University, Columbus; and Miss Mary Hovet, supervisor of mathematics, Ellicott City, Md.

All Regional Directors have been appointed and the following schedule of Re-

gional Orientation Conferences in Mathematics has been approved by the Steering Committee. The issuing of invitations to prospective participants is entirely in the hands of the Regional Directors. The results of these conferences will be summarized in a brochure to be distributed by the NCTM.

Date	Location	Director	Region
Oct. 3–4	Philadelphia, Pa.	Mr. M. Albert Linton, Jr., William Penn Charter School, Philadelphia	Connecticut, Delaware, District of Columbia, Maine, Maryland, Massachusetts, New Hampshire, New Jersey, Pennsylvania, Rhode Island, Vermont.
Oct. 10–11	Iowa City, Ia.	Prof. H. Vernon Price, Univ. of Iowa	Illinois, Iowa, Minnesota, Nebraska, North Dakota, South Dakota, Wisconsin.
Oct. 27–28	Atlanta, Ga.	Mr. H. Mack Huie, Atlanta Board of Education	Alabama, Georgia, Louisiana, Mississippi, North Carolina, South Carolina, Virginia.
Nov. 3–4	Portland, Ore.	Mr. William W. Matson, Mathematics Supervisor, Portland Public Schools	Alaska, Idaho, Montana, Oregon, Washington, Wyoming.
Nov. 18–19	Los Angeles, Calif.	Prof. Clifford Bell, U.C.L.A.	Arizona, California, Hawaii, Nevada, Utah.
Dec. 2–3	Topeka, Kans.	Mrs. Marjorie L. French, Topeka	Arkansas, Colorado, Kansas, Missouri, New Mexico, Oklahoma, Texas.
Dec. 9–10	Miami, Fla.	Mrs. Agnes Y. Rickey, Dade County Public Schools, Miami	Florida, Puerto Rico.
Dec. 15–16	Cincinnati, O.	Miss Mildred Keiffer, Supervisor of Mathematics, Cincinnati Public Schools	Indiana, Kentucky, Michigan, Ohio, Tennessee, West Virginia.

Mathematics and industry

by Marie S. Wilcox, Retiring Chairman, Committee on Co-operation with Industry

Previous articles sponsored by this committee have reported assistance which industries have been giving to students and teachers of mathematics. Herein we should like to consider a way in which schools may assist industry and at the same time assist their own students—by giving additional consideration to the training of technicians.

"In the current emphasis on education for the gifted," according to Kenneth C. Skeen,[1] "we must not overlook the importance of well-trained workers in supporting roles. Our technical creativity depends on trained producers at all levels."

Mr. Skeen teaches in a community college in the San Francisco Bay area where the night school enrollment almost equals the day enrollment. Many of the students enter technical majors, and a large number of evening students are there to improve their training for technical or semitechnical assignments. Mr. Skeen has conferred with men in many positions as he has studied the manpower needs of one hundred industries.

Following are other quotations from Mr. Skeen.

"Probably fifty per cent of our industries are using graduate professional

[1] Address delivered at the Annual Meeting of the National Council of Teachers of Mathematics, Dallas, Texas, April 3, 1959.

people for work that could be done by technicians. It is apparent that the young engineer just out of college can learn about his company by doing some work in drafting, some clerical duties, some general training jobs, but studies throughout the country indicate that this kind of underassignment often continues far beyond a reasonable training period. On the other hand, technical assignments are filled in large numbers by people with less than the desired amount of training. In other words, the professional technical man is often overtrained for the job to which he is assigned and the technician who assists the professional is just as often undertrained."

Mr. Skeen acknowledges that it is difficult to state just what a technician is and equally difficult to define the education and training for a technician. He believes, however, that the technician's job should emphasize technical knowledge and technical skill, require a knowledge of professional work but not necessarily the skill to do it, involve some use of instruments, require effective use of language to interpret orders and make reports, involve the element of leadership in supervisory occupations, require understanding of industrial equipment and processes, and should frequently involve visualization of plans and drawings with a degree of creative design.

Concerning the training of a technician, Mr. Skeen quotes the typical employer as saying that "they should go at least through applied trigonometry, make the algebra strong, have instruction in the use of the slide rule, and have all the mathematics above that level which they can." Other training desired for a technician includes appropriate courses in science, drawing, and human relations, along with laboratory courses in machines, in instrumentation, in electronics, or in other basic specialty, with some understanding of corporate structure. But mathematics is the point of first emphasis in 90 per cent of the cases, with English a fairly close second.

Will a technician trained in this way find employment? The National Association of Manufacturers reports: "Aircraft manufacturing, research, and development are fields loaded with opportunity and challenge for technicians—as tool designers, technical writers, electronics specialists, draftsmen, engineering sales representatives. In fact, 138 different classifications of technicians are needed to build a modern jet fighter. . . .

"Many industries have entry jobs for technicians and have an advancement structure for them. . . . Beginning salaries are good for students with two years of college. High school graduates may enter at lower pay rates."

Report on the 1959 election of officers

The official count of the results of the election for officers and directors of the Council has been completed by the Remington-Rand Corporation. A report of the count has been transmitted to the President of the Council, to the Board of the National Council, and to the Annual Business Meeting.

The official count shows that the following persons were elected:

President:
Phillip S. Jones

Vice-President for Senior High School:
William H. Glenn

Vice-President for Elementary School:
Clarence Ethel Hardgrove

Directors:
J. Houston Banks
Irvin H. Brune
Robert E. K. Rourke

Respectfully submitted,
Mrs. Ida Bernhard Puett, Chairman
Committee on Nominations
and Elections

Annual financial report

by M. H. Ahrendt, Executive Secretary, NCTM, Washington, D. C.

The report below gives a brief picture of the financial condition of the Council as of May 31, 1960. Readers who wish to compare this report with the report for the previous fiscal year will be able to make some interesting comparisons.

Both receipts and expenditures increased, in step with the rapidly growing membership. The fact that your officers were able to operate without increasing expenses in proportion to the increased receipts made it possible for a significant amount to be added to our cash resources, greatly strengthening the financial condition of the Council.

It is interesting to note that the per cent of increase in the sale of publications was more than double the per cent of increase in membership and subscription receipts, thus indicating a growing acceptance on the part of teachers of mathematics of our yearbooks and pamphlets.

Receipts and expenditures of the National Council of Teachers of Mathematics for the fiscal year, June 1, 1959—May 31, 1960

June 1, 1959—Total Cash Resources		$ 99,579.36
RECEIPTS		
Memberships with subscriptions to THE MATHEMATICS TEACHER	$ 86,846.23	
Memberships with subscriptions to *The Arithmetic Teacher*	28,158.80	
Institutional subscriptions to THE MATHEMATICS TEACHER	33,524.84	
Institutional subscriptions to *The Arithmetic Teacher*	30,759.45	
Subscriptions to *The Mathematics Student Journal*	20,088.46	
Sale of advertising space in THE MATHEMATICS TEACHER	13,092.89	
Sale of advertising space in *The Arithmetic Teacher*	3,000.14	
Rental of membership list	1,829.24	
Interest on investments	1,467.77	
Net profit from conventions	3,452.37	
Miscellaneous	679.30	
Sale of publications		
Yearbooks	58,859.80	
Miscellaneous	32,770.01	
Total receipts	$314,529.30	
EXPENDITURES		
Washington office	92,426.40	
Purchase of office equipment	2,511.43	
President's office	5,414.44	
Vice-presidents' office expenses	1,228.84	
THE MATHEMATICS TEACHER	44,107.33	
The Arithmetic Teacher	21,450.53	
The Mathematics Student Journal	11,366.79	
Committee work	2,540.50	
Travel by Board members	6,100.14	
Chicago Policy Conference	2,917.10	
Convention program expenses	968.30	
Contributions to White House and In-service Conferences	1,100.00	
Preparation and printing of yearbooks	29,848.33	
Preparation and printing of supplementary publications	17,323.35	
Storage of shipment of publications, miscellaneous	3,011.42	
Total expenditures	$242,314.90	
Increase in Cash Resources		$ 72,214.40
May 31, 1960—Total Cash Resources		$171,793.76

The 1960 budget

The Budget Committee for 1960 was composed of Dr. Bruce E. Meserve, Dr. H. Vernon Price and Dr. Houston T. Karnes, Chairman.

The Committee met at the Washington Office, January 21–22, 1960. During this period, the Committee had the fine services of the Executive Secretary, M. H. Ahrendt, and the availability of all the important records in the Central Office.

On the basis of the work done during this meeting, a proposed budget was prepared and submitted, by mail, to the Board on March 4, 1960. At the Buffalo Meeting, the Budget was presented by the Chairman and discussed thoroughly by the Board. The Budget as adopted by the Board, which includes a few changes from the Budget as prepared by the Committee, is submitted below:

Receipts		
Memberships (26,000 @ $5.00)	$130,000	
Memberships (2,500 @ $1.50)	3,750	
MSJ subscriptions (70,000 @ 30 cents)	21,000	
Memberships (for second journal: 5,000 @ $3.00)	15,000	
Subscriptions (both journals: 5,000 @ $7.00)	35,000	
Subscriptions (both journals: 5,000 @ $6.75)	33,750	
Advertising in journals	10,000	
Interest on U. S. Treasury Bonds	375	
Interest on saving accounts	2,350	
Total		$251,225
Expenditures		
Washington Office		
Executive Secretary		
Salary	$ 11,275	
Professional assistant		
Salary	7,825	
Editorial assistant		
Salary	5,750	
Secretarial, editorial and clerical help		
Salaries	55,000	
General		
General office expenses	25,000	
Travel	4,000	
Special benefits (Social Security, hospitalization and retirement)	8,500	
Total		$117,350
President's Office		
Office expenses	$ 1,000	
Secretary	4,000	
Travel	2,000	
President's Fund	1,000	
Total		$ 8,000
Vice-Presidents' Office		
Office expenses	$ 800	
Total		$ 800
The Mathematics Teacher		
Printing and mailing	$ 45,000	
Edit, Inc.	3,600	
Secretary	1,000	
Office expenses and equipment	800	
Travel	750	
Total		$ 51,150

The Arithmetic Teacher		
Printing and mailing	$ 22,500	
Edit, Inc.	3,300	
Secretary	2,000	
Office expenses and equipment	1,000	
Travel	750	
Total		$ 29,550
Mathematics Student Journal		
Printing and Mailing	$ 15,000	
Office expenses	200	
Travel	750	
Experimental program	500	
Total		$ 16,450
Other		
Committee expenses	$ 7,925	
Travel: Committees	8,000	
Subcommittee work	5,000	
Travel: Vice-Presidents, Directors, Recording Secretary	5,000	
Convention program expenses	2,000	
Total		$ 27,925
Grand Total		$251,225

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C.

NCTM convention dates

NINETEENTH CHRISTMAS MEETING

December 27-30, 1960
Arizona State University, Tempe, Arizona
Lehi Smith, Arizona State University, Tempe, Arizona

THIRTY-NINTH ANNUAL MEETING

April 5-8, 1961
Conrad Hilton Hotel, Chicago, Illinois
Hobert Sisler, Morton High School West, 2400 Home Avenue, Berwyn, Illinois

JOINT MEETING WITH NEA

June 28, 1961
Atlantic City, New Jersey
M. H. Ahrendt, 1201 Sixteenth Street, N. W., Washington 6, D. C.

TWENTY-FIRST SUMMER MEETING

August 21-23, 1961
University of Toronto, Toronto, Canada
Father John C. Egsgard, St. Michael's College School, Toronto, Canada

Other professional dates

Men's Mathematics Club of Chicago and Metropolitan Area
October 21, 1960
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Men's Mathematics Club of Chicago and Metropolitan Area
November 18, 1960
YMCA Hotel, 826 South Wabash Avenue, Chicago, Illinois
Vernon R. Kent, 1510 South Sixth Avenue, Maywood, Illinois

Northern Section, California Mathematics Council
December 9-11, 1960
Asilomar Fall Conference
Mike Donahoe, P. O. Box 1385, Carmel, California

Minutes of the Annual Business Meeting

Statler-Hilton Hotel, Buffalo, New York
April 22, 1960

Dr. Harold P. Fawcett, President, called the meeting to order at 4:00 P.M.

I. A motion was made, seconded, and passed to approve the minutes of the meeting of April 3, 1959, as printed in the *Journals*.

II. *Report of the Recording Secretary.*

The Board of Directors has held three meetings since the last meeting of the Council in Dallas in 1959. These meetings were as follows: Ann Arbor, Michigan, August 16, 1959; Washington, D. C., December 18–19, 1959; and Buffalo, New York, April 19–20, 1960. In the brief time allowed, it is impossible to relate all of the actions of the Board. Much of the work is routine business, receiving reports from a large number of committees, and in discussing at length the work of the Council. This report merely attempts to inform the membership of the major actions taken.

Ann Arbor Meeting

A. A decision was made at the Dallas meeting to add two new staff members to the Washington Office. These were a professional assistant and an editorial assistant. The persons selected for these new positions were formally announced at the Ann Arbor meeting. They are: Mr. James Brown, professional assistant, and Miss Miriam Goldman, editorial assistant. The addition of these two new staff members has proved to be of great aid in the work of the Council.

B. Dr. E. Glenadine Gibb, of Iowa State Teachers College, was appointed to the position of Editor of *The Arithmetic Teacher*. She succeeds Dr. Ben Sueltz, the first editor, who has done a fine job in developing this outstanding journal. Dr. Sueltz has served the maximum time allowable.

C. Final plans for the Policy Conference were adopted. This Conference will be referred to later.

D. The Board approved submission of a proposal to the National Science Foundation for a grant to sponsor regional meetings with supervisors and administrators on a survey of the present curriculum studies. This proposal has been granted by the National Science Foundation. The program is under the directorship of Frank B. Allen. During the fall of 1960 there are to be eight regional conferences in various parts of the country.

Washington Meeting

At the Dallas meeting, the Board adopted the recommendation of the Problem-Policy Committee to hold a Policy Conference in October, 1959. A detailed report of this action is to be found in the minutes of the meeting of the Council as published in the October, 1959, issues of THE MATHEMATICS TEACHER and *The Arithmetic Teacher*. As a result of this Conference, the Board was called into session on December 18, 1959, in Washington. Most of the work of this meeting centered around the recommendations of the Policy Conference.

A. New Committee Structure:
The standing committees of the National Council of Teachers of Mathematics shall consist of the following:
   1. An Executive Committee
   2. A Committee on Professional Standards and Status
   3. A Committee on Professional Relations
   4. A Committee on Publications
   5. A Committee on Educational Policies and Programming

The Executive Committee shall consist of the president and four members of the Board of Directors selected for one year by the president and approved by the Board. The other four standing committees shall have five members each, appointed for a three-year term by the president with the approval of the Board; except that the initial terms shall be one year for one member, two years for two members, and three years for two members. A committee member may serve no more than two successive terms.

Each committee shall be responsible to the Board of Directors but may appoint such subcommittees as deemed necessary to accomplish the responsibilities delegated to it.

Until the Bylaws are revised, the Executive Committee shall appoint a subcommittee on budget and auditing and on nominations and elections. The Publications Committee shall appoint a subcommittee on yearbook planning and one on supplementary publications. Also the Executive Committee shall, until the Bylaws are revised, consist of the president and two members of the Board.

The president shall act as chairman of the Executive Committee. The president shall each year select from each of the

other committees one member who will serve as chairman.

This organization shall become effective immediately following the annual meeting in 1960 and all committees and representatives now in existence shall be dismissed or incorporated in the new structure.

Each committee is free to establish its method of operation and its policies except those specifically denied it by the Board.

B. Elementary School Mathematics Study:

One of the strongest recommendations of the Policy Conference was to the effect that the Council formulate a plan for a complete study of elementary mathematics. The Board adopted this recommendation and authorized the Executive Committee and the Chairman of the Committee on Educational Policies and Programming to develop the procedure and get the work started.

C. Information-Evaluation Study:

In view of the many different projects extant in the field of curriculum matters and related topics, the Board authorized that a plan be devised whereby information and an evaluation of these projects might be made available to the membership of the Council and to the public concerned. The procedure for this study is to be developed by the Executive Committee and the Chairman of the Committee on Educational Policies and Programming.

D. The Board agreed to co-sponsor with the U. S. Department of Education a conference on In-service Reeducation of Mathematics Teachers. This Conference was held in Washington on March 18, 19, and 20 under the direction of Dr. Kenneth Brown and Mr. Myrl Ahrendt.

BUFFALO MEETING

A. Budget:

It is well known that the Council is one of the most rapidly growing educational organizations. As the Council grows, both income and expenditures naturally increase. The Board adopted a budget of $251,225 for the year 1960–61. Since this budget is being published in the October issues of THE MATHEMATICS TEACHER and *The Arithmetic Teacher*, details will not be given here.

B. Bylaws:

For a number of reasons, the Board decided the Bylaws should be revised. A committee was appointed to study this matter. The Board approved the report of the Committee. Since the proposed revisions must be made available to the membership at least thirty days before a meeting of the Council where changes are to be made in the Bylaws, the suggested changes will not be read at this time.

C. The Board has felt for some time that the Council must provide a system of retirement income, for the paid staff, to be in harmony with other organizations. In keeping with its thinking the Board has agreed to adopt a plan as provided by T.I.A.A.

D. Since the work in the Washington Office has grown to such large proportions, the Board has decided to employ the services of a business management company to make a survey of our present operations and to submit recommendations for improvements should such be warranted.

E. Future Conventions:

Approved to date.

1960 Summer: Salt Lake City, Utah
Christmas: Tempe, Arizona
1961 Spring: Chicago, Illinois
Summer: Toronto, Canada
1962 Spring: San Francisco, California
Summer: Madison, Wisconsin
1963 Spring: Pittsburgh, Pennsylvania
Summer: Eugene, Oregon
1964 Spring: Miami Area, Florida

III. *Report of the President.*

President Fawcett gave the following report:

The report to which you have just listened, summarizing important actions of the Board of Directors during the past year, reflects the increasing activity of the National Council in its continuing effort to improve the teaching of mathematics. Perhaps the most significant event during this period was the Chicago Policy Conference, recommended by the Problem-Policy Committee. You have undoubtedly read the preliminary report of this conference, published in the November, 1959, issue of THE MATHEMATICS TEACHER, and the complete report will appear in the April, 1960, issue. To translate into well-considered action the policies proposed at this conference is one of the major tasks now facing the Board of Directors.

During the year the administrative machinery of the Council has squeaked along more or less satisfactorily. There has scarcely been a day when I have not been grateful to the Board for its wisdom in creating at Dallas a program committee to relieve the president of major responsibility for planning and arranging the program of the annual meeting. The Board is indebted to this committee for the excellence of the program now in process and especially to Glenadine Gibb and Ida May Puett who served as co-chairmen of the committee. To the gratitude of the Board I wish to add my own personal note of thanks.

During the year 26 other committees serving the basic purposes of the Council have been at work in varying degrees. To the 142 men and women who have served as members of these committees, and especially to the respective chairmen, I wish to express my thanks and appreciation. As you will note in the April issue of THE MATHEMATICS TEACHER, the Board has reorganized the committee structure of the

Council, better to serve its expanding interests, and the role of the present committees within the new structure is yet to be determined.

Representatives of the National Council sit on boards, commissions and committees of other professional organizations, and to the eight men and women now serving in this capacity we are indeed grateful.

The Council is to be congratulated on the quality of its three *Journals*. Grateful readers from time to time testify to the excellence of these publications and the respective editors are to be highly commended for the quality of their services. It is appropriate that Dr. Ben Sueltz be specifically mentioned since he is now retiring as editor of *The Arithmetic Teacher*. Six years ago, at the request of the Board, he launched this publication, and its steady growth is silent testimony to his creative genius. As its only editor, he has now completed the full complement of time allowed under our Bylaws. The new editor is Dr. Glenadine Gibb, and our best wishes are with her as she adds this additional responsibility to her busy professional life.

Included in the budget for the past year was an item to provide a half-time secretary for the president. This increased secretarial service, along with relief from program planning, made it possible to meet the minimum obligations of the president's office, but in my judgment that is not enough, for to be satisfied with the minimum is to approve a static position which is certainly untenable. We expect the president to keep the machinery running, but we also expect him to provide a quality of creative leadership which will indeed be instrumental in "promoting the interests of mathematics in America, especially in the elementary and secondary fields." The opportunities for service are almost unlimited, and he could, in fact, easily invest his full time in professional activities directly related to the president's office. In the usual situation, however, he is carrying a full load of responsibility in the institution with which he is associated, and under such circumstances one might wonder whether the best interests of the Council are likely to be served, especially the leadership function. Faced with pressures which make demands beyond human strength, man is likely to neglect those obligations which speak most softly. Is the time spent in class preparation really necessary? Why all this study? Go to class unprepared. Who is there to know the difference? And so through my own limitations, an organization dedicated to the improvement of mathematics education is indirectly responsible for lowering the quality of classroom performance. Even so, I could do little better than keep the machinery running, and, while I am quite willing to concede my own inefficiencies, it is my considered judgment that there is a problem associated with this office which calls for thoughtful study. With a half-time secretary the clerical and stenographic needs of the office *can* be met. Perhaps a full-time secretary with a flair for administrative responsibility could assist the president in keeping the machinery running and thus release him for more productive leadership. To reimburse the institution with which he is associated for half his time is another possibility which seems to have considerable merit.

The major responsibility of The National Council of Teachers of Mathematics is to provide its membership with a quality of service which will enrich the teaching of mathematics in the classrooms of America. It is that purpose which gives direction and emphasis to the programs of our national meetings, and it is to that end that our publications program is directed. I wish now to propose that we increase our service to Council members through providing them from time to time without cost such supplementary publications as selected monographs of general value, reprints of important articles on mathematics education, and the like. The distribution of "The New World of Mathematics" approximately one year ago has undoubtedly stirred many mathematics teachers into a program of professional improvement, and students in many classrooms are for the first time *seeing* mathematics with both physical and intellectual eyes because their teachers received a copy of "A Guide to the Use and Procurement of Teaching Aids for Mathematics" prepared by Emil Berger and Donovan Johnson. To provide such service is to build good will for the Council, to nourish an appetite for comparable publications, and to help some teachers take the first long step in a professional program of self-improvement.

I am impressed by the fact that last year our membership increased by 22%, and I am further impressed by the fact that in the official year now drawing to a close approximately 8,000 new names have been added to our membership rolls, an increase of about 31%. But why is that a source of rejoicing? Is it because more pages will now be needed to print our Membership Directory? Is it because our treasury has been increased by close to $40,000? Is size a criterion of value? These outcomes are but vain and empty symbols. We rejoice because the addition of 8,000 new names to the membership rolls means that at least one of our journals is going into homes of 8,000 teachers who never before were under the direct influence of the Council. It means that 8,000 teachers have taken the initial step on the road to professional improvement, and it means that 8,000 good teachers will undoubtedly become 8,000 better teachers. The growth of our membership means that the influence of the Council reaches into an increasing number of mathematics classrooms; the final test of our services is the extent to which the Council improves teaching competence. Let us make it rich and productive.

When I was a small boy in a Canadian school I one day drew a picture of a horse. Since no one knew what it symbolized, I felt compelled to label it, but that in no way made it look any more like a horse. I note from the program that we are now participating in the "Annual Business Meeting" of the National Council of Teachers of Mathematics, but without that label I

wonder if it would be recognized as such. Crowded into one hour at the close of a busy day are four scheduled reports and an unknown number of Special Committee Reports. Following a consideration of these $4+n$ reports the remaining time is devoted to "resolutions" and "New Business." Is there anyone in the audience who can suggest a more unpopular time for the introduction of "new business"? Is it likely that under these circumstances, colored by the attractive shadow of an impending banquet, even the "old business" will receive the thoughtful consideration it merits? What are the purposes of this "Annual Business Meeting"? Whatever they are, I cannot believe that they are well served through present procedures which attract such a thin attendance. My own thoughts concerning this problem have yielded no satisfactory solution. It is indeed time for a change.

IV. *Report of the Executive Secretary.*

Mr. M. H. Ahrendt, Executive Secretary, gave the following report.

A. Washington Office:

The full staff of the Washington office during this school year has numbered 18 persons. Two professional staff persons were added with the approval of the Board of Directors: J. James Brown, professional assistant, and Miriam Goldman, editorial assistant. A new staff position of administrative assistant was created to help with correspondence related to inquiries and complaints of a business nature. The establishment of this position has enabled us to give much better service to the persons who complain about nondelivery of their materials by the post office and occasional errors made by our office staff. The increase in membership and our growing sales program required the employment of several additional clerical workers.

The office again outgrew its allotted space, and we were forced to move part of our staff to a room on another floor of the NEA Building. The secondary office houses our Billing and Order Section. An intercom system was added to our office telephones to make it easy for the two parts of the office to keep in touch with each other.

The understanding attitude of the Board of Directors in making available some money for the purchase of office equipment has been much appreciated by the entire staff. We have been able to purchase during the year a postal mailing machine, an improved postal mailing scale, a photocopy machine, and several electric typewriters.

The creation of the position of professional assistant has considerably lightened the load of the executive secretary and has enabled the office to give better service in some areas.

B. Membership:

Membership of the Council is experiencing an increase during this year alone which is approximately equal to our total membership only ten years ago. The total members and subscribers should reach approximately 32,000 by May 1, 1960. The membership growth during each of the past four years has been, respectively, 17 per cent, 20 per cent, 22 per cent, and 31 per cent.

C. Financial Condition:

The financial condition of the Council is very satisfactory. The increase in membership and subscription receipts is almost directly proportional to the increase in number of members, or about 31 per cent. However, the increase in income from the sale of publications is about 70 per cent, indicating a growing interest among teachers in our publications. The sale of yearbooks accounts for about two-thirds of the income from our publication sales. At present time, it appears that the total income during the fiscal year from memberships, subscriptions, and sale of publications may reach $325,000.

D. Circulation of Journals:

The circulation of each of the three journals has increased steadily, in step with the growing membership. Below are the print orders for the issues that are now on the press: MATHEMATICS TEACHER, April 1960, 27,000 copies; *Arithmetic Teacher*, April 1960, 14,500 copies; *Mathematics Student Journal*, March 1960, 75,000 copies.

E. New Publications:

The chief new publication for this year has been the 25th Yearbook, *Instruction in Arithmetic*. The number of copies printed was 10,000. The book is off the press and copies have been shipped to Buffalo for sale at the 38th Annual Meeting. The book has been advertised in the journals and in the program for the 38th Annual Meeting, with the statement that copies will be delivered after the presentation of the book at the meeting. The editing of the book was made much easier by having an editorial assistant in the Washington office.

The pamphlet *Five Little Stories*, by W. W. Strader, came off the press recently. The number of copies printed was 6,400. Other pamphlets added to our list of publications are the following: *Mathematics Tests Available in the United States*, 10,000 copies; *New Developments in Secondary School Mathematics*, 3,000 copies; *Mathematics for the Academically Talented Student in the Secondary School*, 9,000 copies.

On the press at the present time are the two bibliographies, *The High School Mathematics Library* and *The Elementary School Mathematics Library*. By a previous

action of the Board of Directors, copies of these two bibliographies are to be mailed free to members of the Council, those members receiving THE MATHEMATICS TEACHER to get the first bibliography and those who take *The Arithmetic Teacher* to receive the second.

Also on the press at the present time is the pamphlet *Notes on Vectors, Analytics, and Calculus for Twelfth-Year Mathematics*, by Abraham W. Glicksman, and the *Number Story*, by Herta T. and Arthur E. Freitag. In the files are two additional manuscripts which will be prepared for the press within the very near future. Their tentative titles are: *A Look at Modern Mathematics*, by Lyman C. Peck, and *What Is Number Theory*, by I. A. Barnett.

Publications reprinted: The following publications have been reprinted since the report of the Executive Secretary was made a year ago: 23rd Yearbook, 5,000 copies; 24th Yearbook, 10,000 copies; *How to Study Mathematics* (on the press), 10,000 copies; *How to Use Your Bulletin Board*, 5,000 copies; *Paper Folding for the Mathematics Class*, 7,000 copies; *How to Use Your Library in Mathematics*, 7,000 copies; *A Guide to the Use and Procurement of Teaching Aids for Mathematics*, 15,000 copies.

V. *Report of the 1960 Committee on Nominations and Elections.*

This report was presented by Mrs. Ida Bernhard Puett, Chairman of the Committee. It is reprinted elsewhere in the October issue and is not repeated here.

VI. *Resolutions.*

Dr. Ben A. Sueltz presented the following resolution:

WHEREAS the National Defense Education Act of 1958 has been instrumental in assisting many schools to obtain valuable equipment for the teaching of mathematics, and

WHEREAS less than one-fifth of the money expended under the provisions of Title III has been used for the purchase of equipment for mathematics, and

WHEREAS typewriters with mathematical characters would be of great usefulness to the teacher in the preparation of tests, manuscripts, and other materials in mathematics,

BE IT RESOLVED that the members of the National Council of Teachers of Mathematics respectfully request the Commissioner of Education to seek a broader interpretation of the provisions of Title III to permit the purchase with funds made available through the Act of typewriters equipped with special mathematical characters.

Following the reading of this resolution, Dr. Sueltz moved its adoption. The motion was duly seconded and passed.

VII. *Report of the Resolutions Committee*

The Resolutions Committee was composed of Miss Annie John Williams and Dr. Clifford Bell. Miss Williams gave the report of the Committee which follows:

> We, the members of the National Council of Teachers of Mathematics, wish to express our sincere appreciation for the hospitality extended to us by the Association of Mathematics Teachers of New York State and the Buffalo Public Schools. We especially wish to thank Louis F. Scholl and his co-chairman, Theresa L. Podmele, and the members of their various committees for the efficient manner in which they are carrying out the careful plans made for the convention. This has done much toward making the meeting a very successful one.
>
> Your city lives up to our expectations and many of us have enjoyed Niagara Falls and the other wonderful tours you have arranged for us.
>
> We also wish to thank the press, the radio and television stations which have co-operated so generously.

After reading the report, Miss Williams moved its adoption. The motion was duly seconded and passed.

VIII. Dr. H. C. Christofferson, former president, moved that the President and the Board be extended a vote of thanks and complimented highly on the outstanding work that was accomplished during the past year. In addition, he moved that Dr. Ben A. Sueltz, retiring Editor of *The Arithmetic Teacher*, be given a vote of praise for the outstanding job he did in developing this fine journal as its first editor. (Secretary's Note: Dr. Sueltz served two terms as Editor, which is the maximum time allowed under our Bylaws.)

The motion was duly seconded and passed.

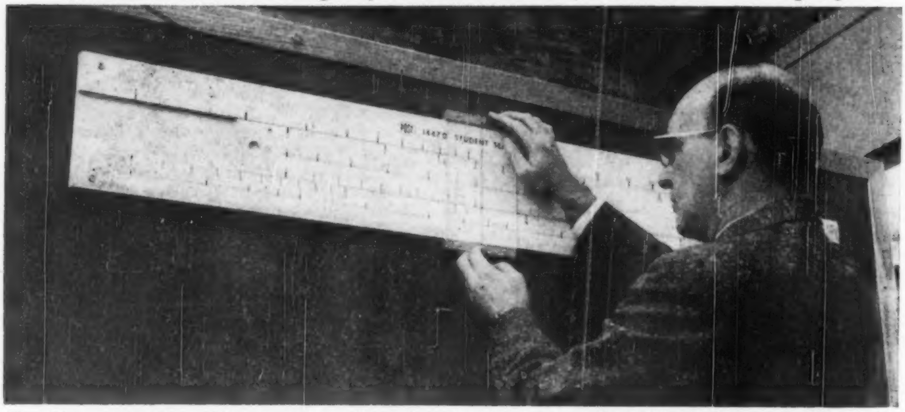
IX. *Adjournment.*

The meeting was duly adjourned at 5:10 P.M.

Respectfully submitted,
Houston T. Karnes
Recording Secretary

The facts about "Bargain" slide rules

by **ROBERT JONES,** *Manager of Educational Sales, Frederick Post Company*



Teaching suggestion: Many math and engineering instructors find this scaled-up model of the POST Student Slide Rule an efficient teaching aid

Unless your students know what to look for in a slide rule, they will be tempted by "bargain" sales messages to buy one of the "less expensive rules" retailing for 75c to $1.25. Here are the short-comings that are certain to show up in these so-called "specials".

- **Markings.** The printed scales on cheap slide rules are frequently inaccurate and *always temporary.* After short periods of use the graduations and numbers fade or rub off and the painted or varnished base wood wears rapidly.
- **Materials.** Because they're made of low cost materials, bargain slide rules often warp as a result of climate changes, the readings become undependable. Slides either stick or become loose. Cursors do not operate properly.

A "bargain" like this is no bargain at all.

You'll be doing a real service for your students if you call their attention to the POST Student Slide Rule. This 10″ Mannheim-type rule features "engine-divided" scales like those on expensive professional rules. Each graduation is *cut* into the white celluloid face so it's a permanent part of the rule. Scales retain their legibility and accuracy for a lifetime of use.

Laminated bamboo construction

Like all POST rules, the Student Slide Rule is made of properly seasoned, laminated bamboo. It cannot warp or shrink, is never affected by humidity changes or atmospheric conditions.

The cursor is framed in metal for durability, and a tension spring maintains the vertical position of the hairline. Either with a cardboard slip cap case at the school price of $2.81 or in a durable black fabricord case at $3.75 ea. The easy-to-understand instruction book often serves as a classroom text. POST 1447 Student Slide Rule will serve the student dependably throughout his school years and on through his adult life.

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Send your slide rule order direct to me at: Educational Sales Division, Frederick Post Company, 3650 North Avondale Avenue, Chicago 18, Illinois. Also tell us if you would like to know about the new classroom Slide Rule Demonstrator.

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